Primordial Non-Gaussianities from Inflation and Associated Soft Theorems

Souradeep Das

Indian Institute of Science

UP400 Presentation

Bachelor of Science (Research)

Supervisor: Dr Rajeev Kumar Jain

Department of Physics

Indian Institute of Science

Date: 22 April 2024

Outline of the Talk

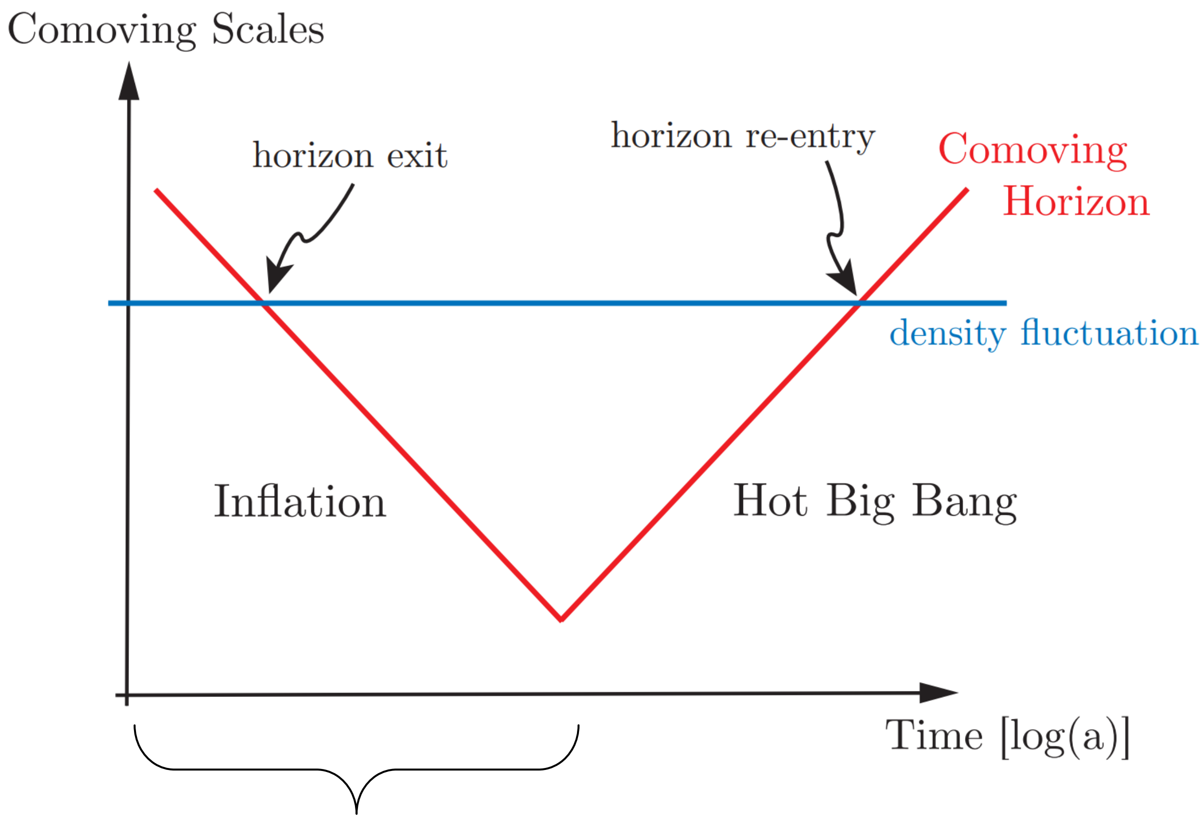
- Introduction and Motivation
  - Background Inflationary Dynamics
  - Evolution of Perturbations
  - Bispectrum and non-Gaussianities
- Outline of the Problem – non-Gaussianities with spectators
- Calculation of the Bispectrum using in-in formalism
- Results
- Conclusion and Outlook

Introduction and Motivation

Inflation: Period of Shrinking Hubble Horizon

Hubble horizon: Causally connected patch in the Universe

$$r_c := \frac{1}{aH}$$



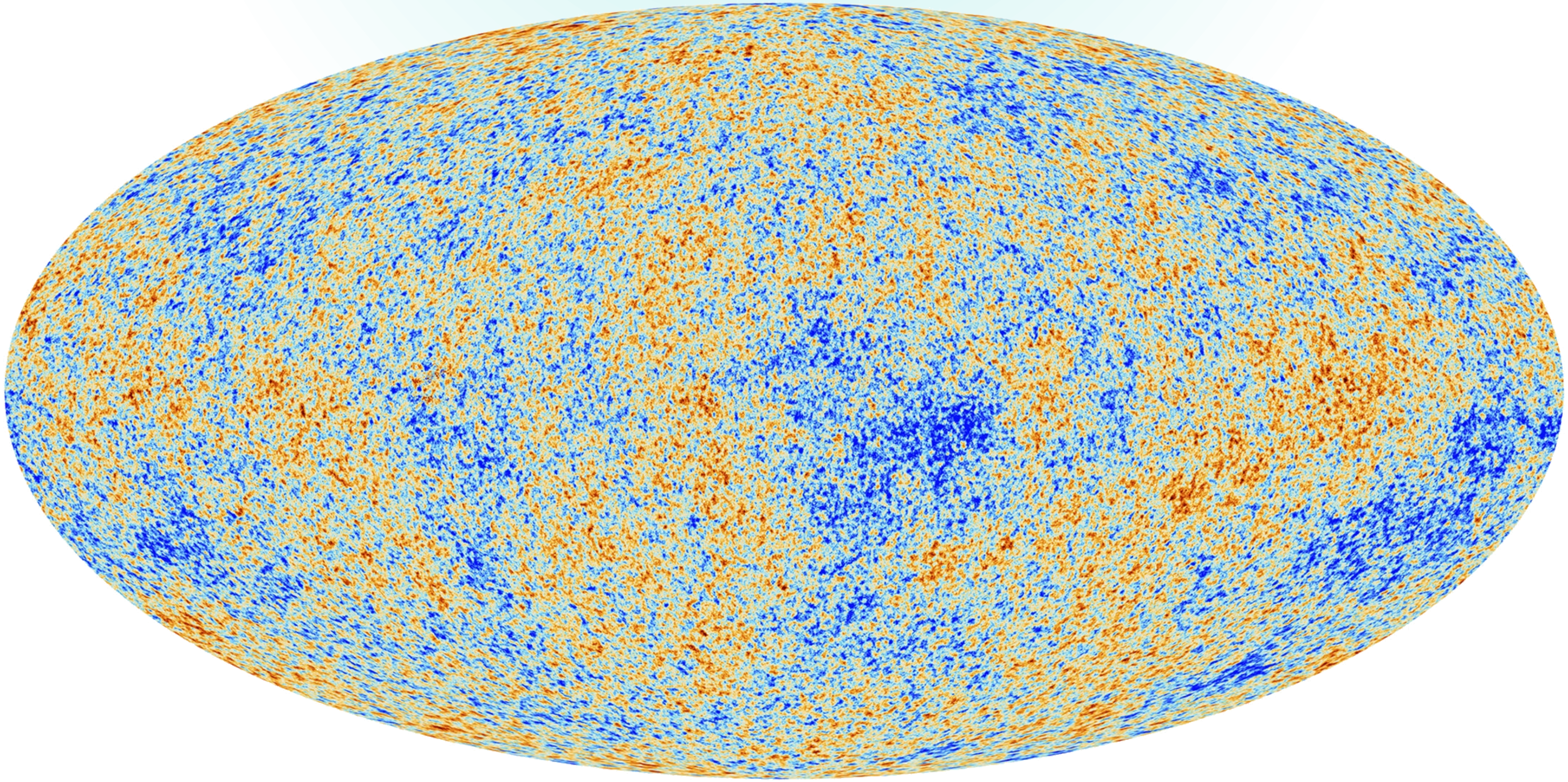
Dominated by **Inflaton**

Cosmological Perturbations

- Departure from homogeneous and isotropic, flat FRW Universe: $\mathrm{d}s^2 = a^2(\tau)\left(-\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)$
- Scalar: Comoving curvature perturbation : Intrinsic curvature of constant-time hypersurfaces
- Tensor: - Anisotropic part of spatial metric

$$\mathrm{d}s^2 = a^2\left(-\mathrm{d}\tau^2 + e^{2\zeta}\,[e^{\gamma}]_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)$$

Density Fluctuations give rise to CMB Anisotropies



Source: Planck (2018)

Evolution of Perturbations in Free Theory

- Primordial perturbations are quantum fields
- Free theory of slow-roll inflation– 2nd order action in or
- Solve for mode fun

$$\zeta_k(\tau) = \frac{1}{2\sqrt{\varepsilon}} \frac{iH}{k^{3/2}} (1 + ik\tau) e^{-ik\tau}$$

$$\gamma_k(\tau) = \frac{iH}{k^{3/2}} (1 + ik\tau) e^{-ik\tau}$$

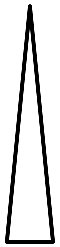
- We use Bunch-Davies initial condition

The Bispectrum

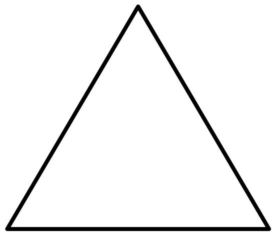
- Bispectrum: 3-point correlation function of field fluctuations

$$\langle Q_1(t,\mathbf{k_1})Q_2(t,\mathbf{k_2})Q_3(t,\mathbf{k_3})\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B_{Q_1Q_2Q_3}(k_1,k_2,k_3)$$

  - Three momenta must form a triangle
- Squeezed limit of triangle:
  - Semi-classical argument – bispectrum in terms of power-spectrum



Squeezed limit

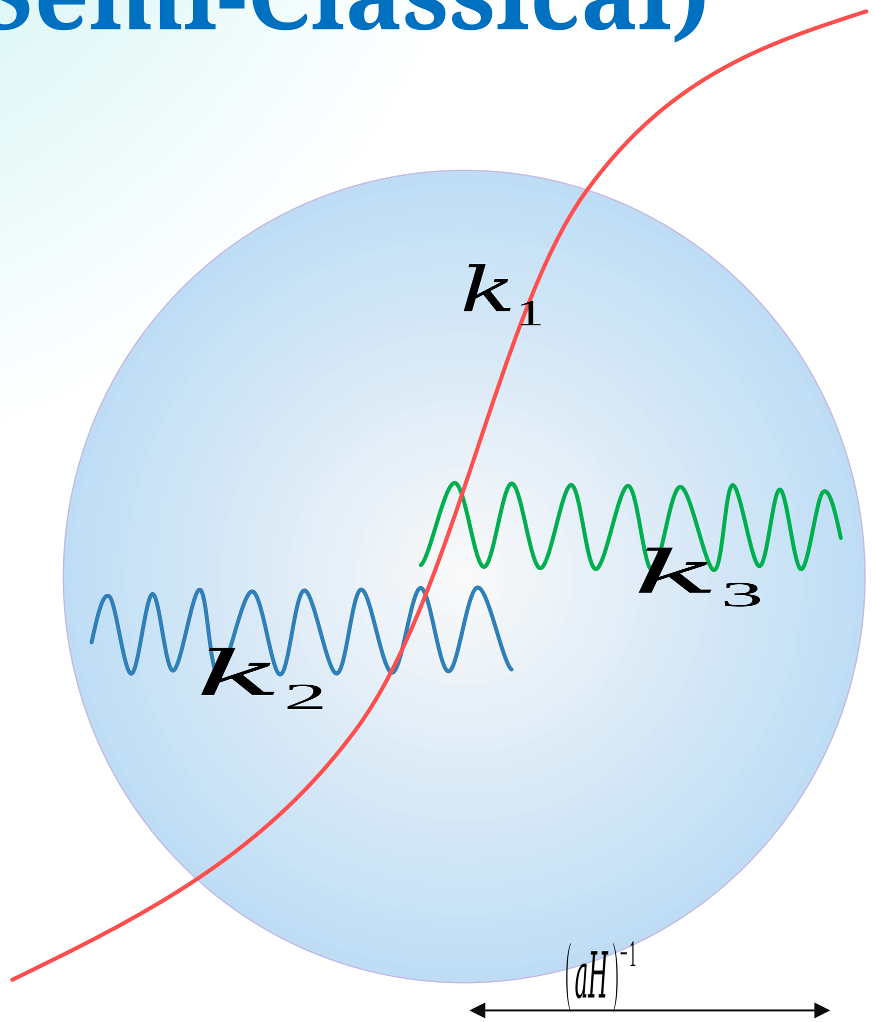


Equilateral limit

Flattened limit

Background Wave Method (Semi-Classical)

- Squeezed limit of bispectrum , *i.e.*,
- Long-wavelength mode absorbed by redefining the coordinates.
- In the modified coordinates, calculate and correlate with the long-wavelength fluctuation   [Maldacena, 2003]



Maldacena Consistency Relation

- Result:
  - **Maldacena Consistency Relation**
- Consider another scalar field
- For ,
- For ,

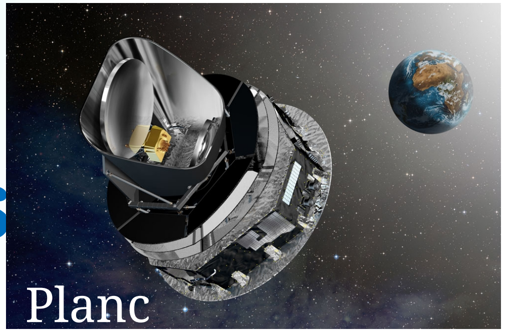
$$\Delta_\zeta^2(k) = \frac{k^3}{2\pi^2} P_\zeta(k)$$

(Squeezed limit of or)

Non-Gaussianities and the Bispectrum

- Statistics of fluctuations:
  - Almost perfect Gaussian distribution
  - Small deviations – non-Gaussianities
  - Can be primordial – produced during inflation
- Non-Gaussianities – always captured in higher correlation functions (bispectrum, trispectrum etc.)
- We are interested in the bispectrum of fluctuations!

Primordial Non-Gaussianities



- Primordial non-Gaussianities – produced due to quantum interactions between fields during inflation
- Observed in CMB! (Planck 2018)
  - Hints at non-trivial interactions in the inflationary epoch – add extra fields!
- Primordial Non-Gaussianities predicted in:
  - Curvatons (Bartolo *et al*, 2004; Sasaki *et al*, 2006)
  - Vector fields (Shiraishi *et al*, 2012)
  - Primordial magnetic fields (Jain and Sloth, 2012, 2013)

Shape	f_{NL}
Local	-0.9 ± 5.1
Equilateral	-26 ± 47
Orthogonal	-38 ± 24

Outline of The Problem

Model Spectator Field

- Light scalar field , non-minimal derivative coupling with [Sai and Jain, 2023]
- Small mass and low energy density – negligible effect on inflationary dynamics
- Direct dilatonic coupling:

minimal coupling | non-minimal coupling | derivative coupling

Evolution of Spectator Field

- Free-theory: 2nd order action for

$$S_\sigma = -\left(1 + 3\alpha H^2\right) \int \mathrm{d}^4x \, a^2 \lambda \frac{1}{2}\left[(\partial\sigma)^2 - \sigma'^2 + \left(\tilde{m}^2 a^2 + \tilde{\xi}\frac{a''}{a}\right)\sigma^2\right]$$

- Mode function: $\sigma_k(\tau) = \frac{1}{\sqrt{1 + 3\alpha H^2}} \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \frac{\sqrt{-\tau}}{a\sqrt{\lambda}} H_\nu^{(1)}(-k\tau)$

$$\nu = \sqrt{\left(n + \frac{3}{2}\right)^2 - 2\tilde{\xi} - \frac{\tilde{m}^2}{H^2}}$$

- Power spectrum:

$$P_\sigma(k) \propto k^{-2\nu}$$

$\tilde{m}^2 = \frac{m^2}{1+3\alpha H^2}$ and $\tilde{\xi} = \frac{\xi}{1+3\alpha H^2}$

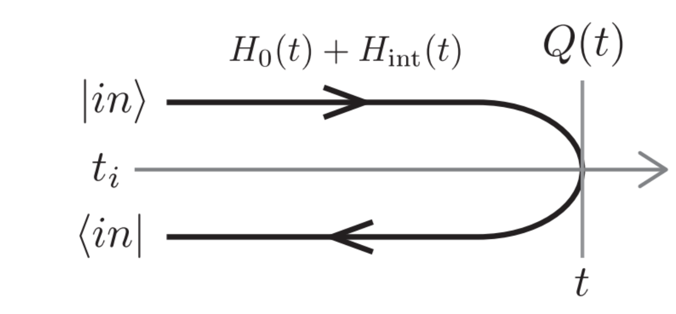
Aim

- Aim: To obtain the full bispectrum of this field perturbations with metric perturbations:

- To verify the consistency relations in the squeezed limit

Techniques Used

In-in Formalism: Master Formula

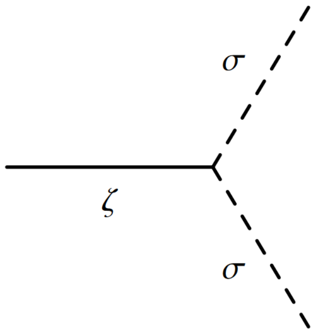


- Master formula:

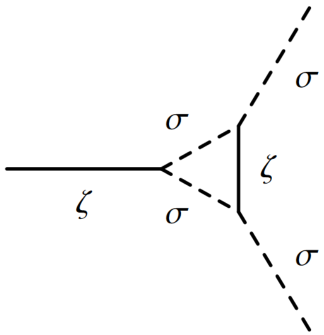
$$\langle Q(t)\rangle = \left\langle 0\middle|\bar{T}\left(e^{i\int_{-\infty(1-i\epsilon)}^{t} \mathrm{d}t' H_I^{int}(t')}\right)Q_I(t)T\left(e^{-i\int_{-\infty(1+i\epsilon)}^{t} \mathrm{d}t'' H_I^{int}(t'')}\right)\middle|0\right\rangle$$

  - Tree-level term:

$$\langle Q(t)\rangle^{(1)} = -i\int_{-\infty(1-i\epsilon)}^{t} \mathrm{d}t' \langle 0|\left[Q_I(t), H_I^{int}(t')\right]|0\rangle$$



(a) Tree-level diagram



(b) Lowest-order loop-level diagram

Results

Scalar Cross-Correlation: Bispectrum of

- Interaction Hamiltc $H_{\zeta\sigma\sigma} = -\frac{1}{2}\int d^3x\, a^2\lambda'(\tau)\tau\zeta\left(\sigma'^2 - (\partial\sigma)^2 - \left(a^2\tilde{m}^2 + \tilde{\xi}\frac{a''}{a}\right)\sigma^2\right)$

- Bispectr

$$\mathcal{B}_{\zeta\sigma\sigma}(k_1,k_2,k_3) = \frac{2n}{1+3\alpha H^2}|\zeta_{k_1}^{(0)}(\tau_I)|^2|\sigma_{k_2}^{(0)}(\tau_I)||\sigma_{k_3}^{(0)}(\tau_I)|$$
$$\times\left[-k_2k_3\tilde{\mathcal{I}}_\nu^{(1)} + \frac{1}{2}\left(k_2^2+k_3^2-k_1^2\right)\tilde{\mathcal{I}}_\nu^{(2)} + (n+3/2-\nu)\left(2\nu\tilde{\mathcal{I}}_\nu^{(3)} + k_2k_3\tilde{\mathcal{I}}_\nu^{(4)}\right)\right]$$

$$\nu = \sqrt{\left(n+\frac{3}{2}\right)^2 - 2\tilde{\xi} - \frac{\tilde{m}^2}{H^2}} = n + 3/2 - \delta$$

$\tilde{\mathcal{I}}_\nu^{(1)}$ $\tilde{\mathcal{I}}_\nu^{(4)}$

- … are time-integrals of mode functions, resulting in functions of , and , depending on parameter
- Define non-linearit $\mathcal{B}_{\zeta\sigma\sigma}(k_1,k_2,k_3) = \frac{1}{2}b_{NL}P_\zeta(k_1,\tau_I)(P_\sigma(k_2,\tau_I)+P_\sigma(k_3,\tau_I))$ f only ratios of sides)

Divergence in

- Power-law divergence : in the limit appears in

$$\tilde{I}_\nu^{(1)}, \text{ when } \nu > 7/2$$

$$\tilde{I}_\nu^{(2)}, \text{ when } \nu > 5/2$$

$$\tilde{I}_\nu^{(3)}, \text{ when } \nu > 3/2$$

$$\tilde{I}_\nu^{(4)}, \text{ when } \nu > 5/2$$

Relevant only when

- Physicality of correlation: Divergence properties of the integrals $\tilde{I}_\nu^{(1)} \ldots \tilde{I}_\nu^{(4)}$ limit the values of (or)

Case I:
Choose
Case II:
Choose only

Case I:

When $n = -1$, $\nu = 1/2$:

$$\tilde{I}_\nu^{(1)} = -\frac{2k_1 + k_2 + k_3}{(k_2 k_3)^{1/2} k_t^2}$$
$$\tilde{I}_\nu^{(2)} = \frac{2k_1 + k_2 + k_3}{(k_2 k_3)^{1/2} k_t^2}$$
$$\tilde{I}_\nu^{(3)} = -\frac{-k_1 + (\gamma + \log(-k_t \tau_I))(k_2 + k_3)}{(k_2 k_3)^{1/2}}$$
$$\tilde{I}_\nu^{(4)} = \frac{\pi(k_2 + k_3)}{2(k_2 k_3)^{3/2}}$$

When $n = 0$, $\nu = 3/2$:

$$\tilde{I}_\nu^{(1)} = \frac{2k_1 + k_2 + k_3}{(k_2 k_3)^{1/2} k_t^2}$$
$$\tilde{I}_\nu^{(2)} = \frac{k_1^3 + 2k_1^2(k_2 + k_3) + (2k_1 + k_2 + k_3)(k_2^2 + k_2 k_3 + k_3^2)}{(k_2 k_3)^{3/2} k_t^2}$$
$$\tilde{I}_\nu^{(3)} = \frac{(k_1^2 + k_2 k_3)(k_2 + k_3) + k_1(k_2^2 - k_2 k_3 + k_3^2)}{3(k_2 k_3)^{3/2}} - \frac{(k_1^3 + k_2^3 + k_3^3)(-1 + \gamma + \log(-k_t \tau_I))}{3(k_2 k_3)^{3/2}}$$
$$\tilde{I}_\nu^{(4)} = -\frac{2}{\tau_I (k_2 k_3)^{3/2}}$$

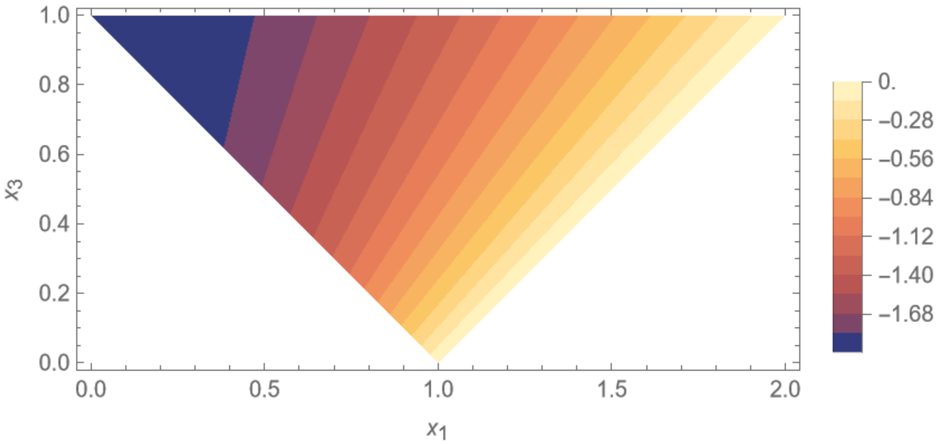
When $n = 1$, $\nu = 5/2$:

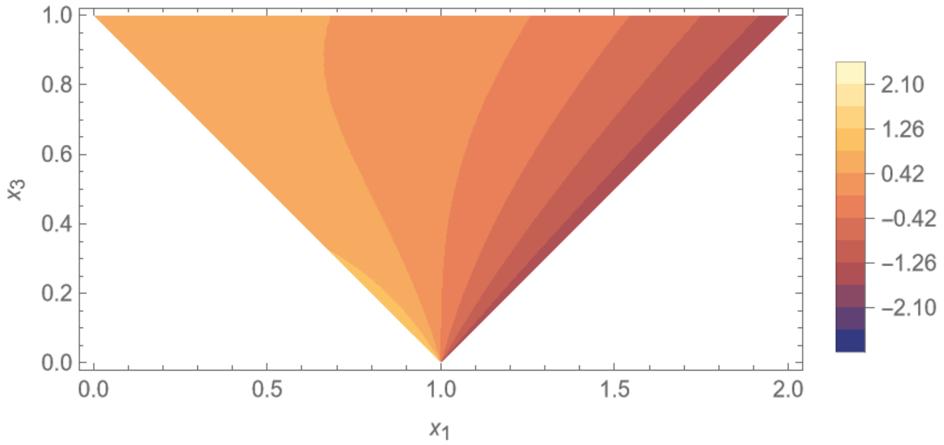
$$\tilde{I}_\nu^{(1)} = \frac{k_1^3 + 2k_1^2(k_2 + k_3) + (2k_1 + k_2 + k_3)(k_2^2 + k_2 k_3 + k_3^2)}{(k_2 k_3)^{3/2} k_t^2}$$
$$\tilde{I}_\nu^{(2)} = \frac{(2k_1 + k_2 + k_3)\left(\frac{k_2^5 - k_3^5}{k_2 - k_3}\right) + k_1^2(3k_t^3 + k_2^3 + k_3^3 - 3k_t k_2 k_3)}{(k_2 k_3)^{5/2} k_t^2} - \frac{3k_1^3(\gamma + \log(-k_t \tau_I))}{(k_2 k_3)^{5/2}}$$
$$\tilde{I}_\nu^{(3)} = \frac{9k_1^3}{10\tau_I^2 (k_2 k_3)^{5/2}}$$
$$\tilde{I}_\nu^{(4)} = \frac{-2}{\tau_I^3 (k_2 k_3)^{5/2}}$$

Plots of for

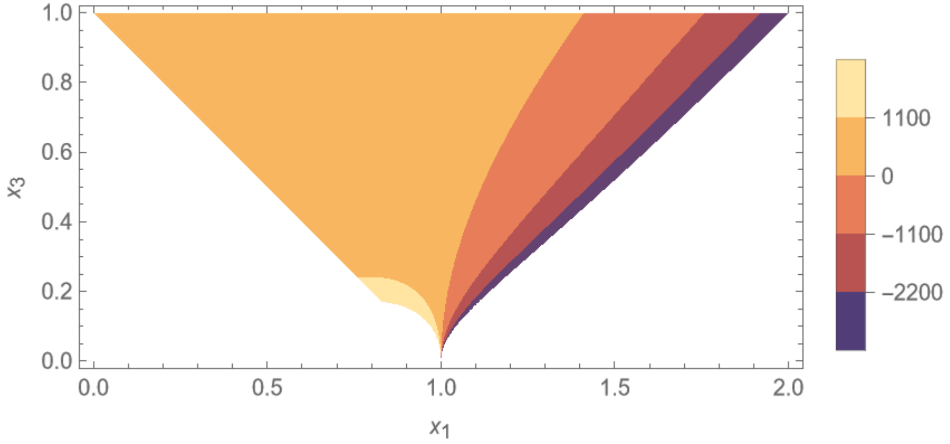
Case I:

Axes:





Squeezed limit Equilateral limit Flattened limit



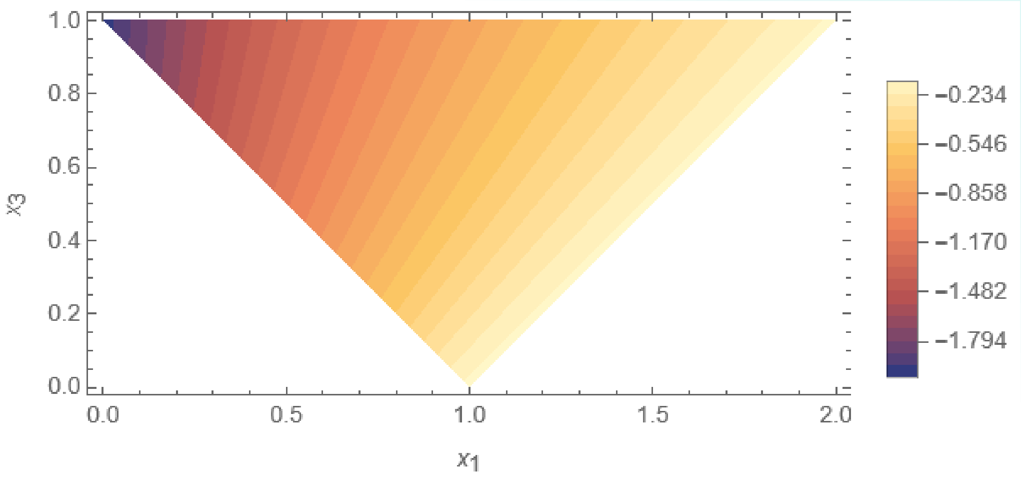
When

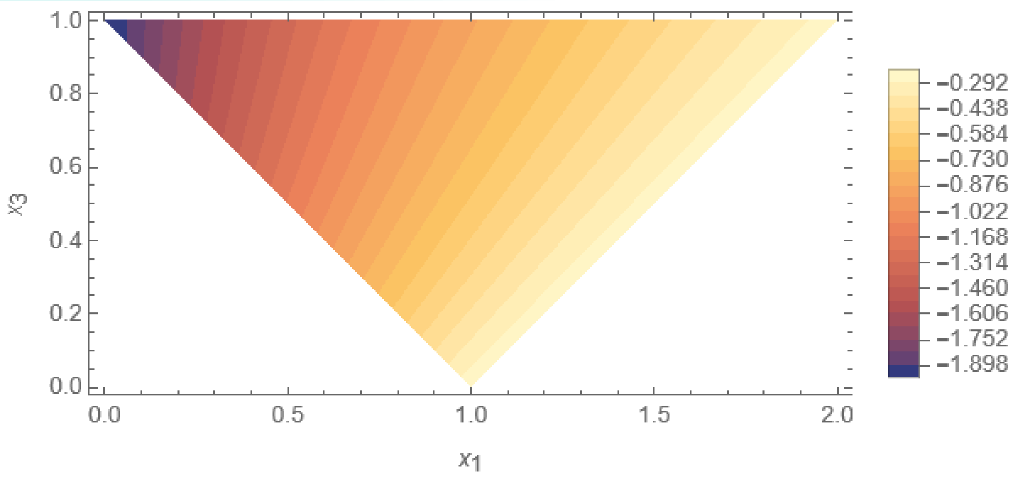
- Hankel functions of non-half-integer index : do not have analytic form
- Series expansion to evaluate integrals (from to , where)
  - Problem: Can only integrate on finite interval of conformal time
  - Solution: Choose a finite interval of integration; impose analytically evaluated condition on squeezed limit
  - Take integral from to , where
  - Reasoning: The mode functions oscillate rapidly in sub-horizon regime; hence time-integral must cancel out over large intervals.

Plots of for

Case II:

Axes:





δ	Squeezed ($x_1 \to 0,\ x_3 \to 1$)	Squeezed* ($x_1 \to 1,\ x_3 \to 0$)	Flattened ($x_1 \to 2,\ x_3 \to 1$)	Equilateral ($x_1 \to 1,\ x_3 \to 1$)
0	-2	0	0	-1.333
0.0005	-2	-0.118	-0.133	-0.820
0.001	-2	-0.235	-0.249	-0.889

Tensor Cross-Correlation: Bispectrum of

- Bispectrum obtained from Hamiltonian at order :

$$H_{\gamma\sigma\sigma} = -\frac{1}{2}(1+5\alpha H^2)\int \mathrm{d}^3x\, a^2\lambda(\phi)\gamma_{ij}\partial_i\sigma\partial_j\sigma$$

- Bispectrum:

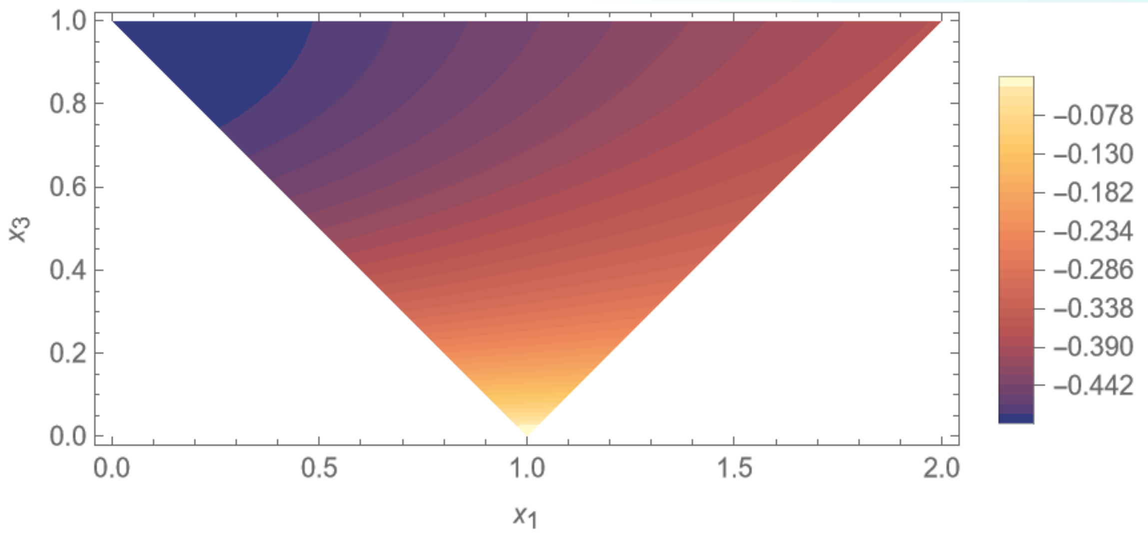
$$B_{\gamma\sigma\sigma}(k_1,k_2,k_3) = -\epsilon_{ij}k_{2,i}k_{3,j}\left(\frac{1+5\alpha H^2}{1+3\alpha H^2}\right)\left|\gamma^{(0)}_{k_1}(\tau_I)\right|^2|\sigma^{(0)}_{k_2}(\tau_I)||\sigma^{(0)}_{k_3}(\tau_I)|\tilde{\mathcal{I}}^{(2)}_{\nu}$$

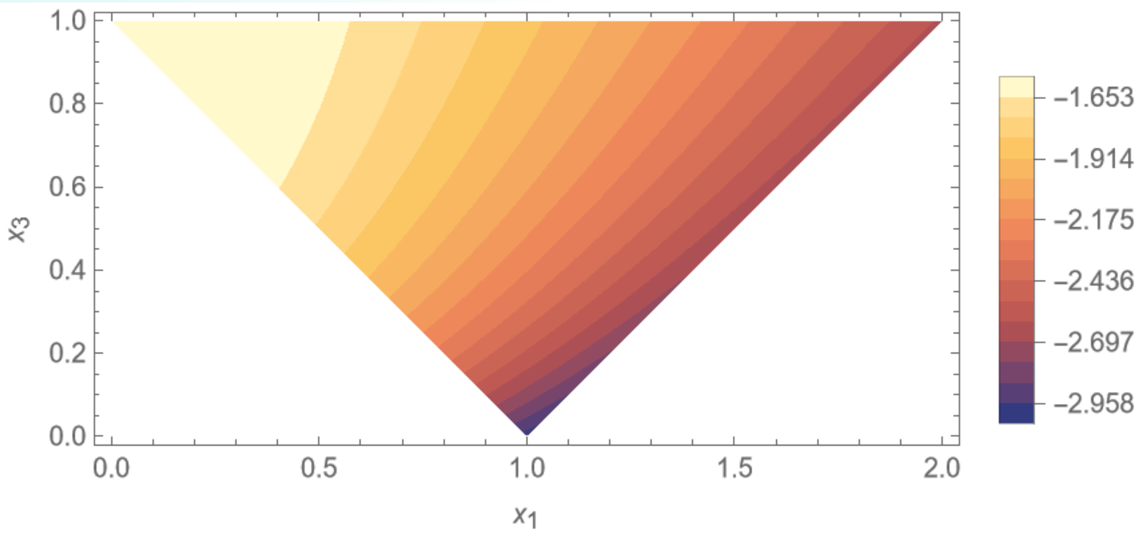
- Divergence occurs only for
- Define similarly for this correlation

Plots of for

Case I:

Axes:

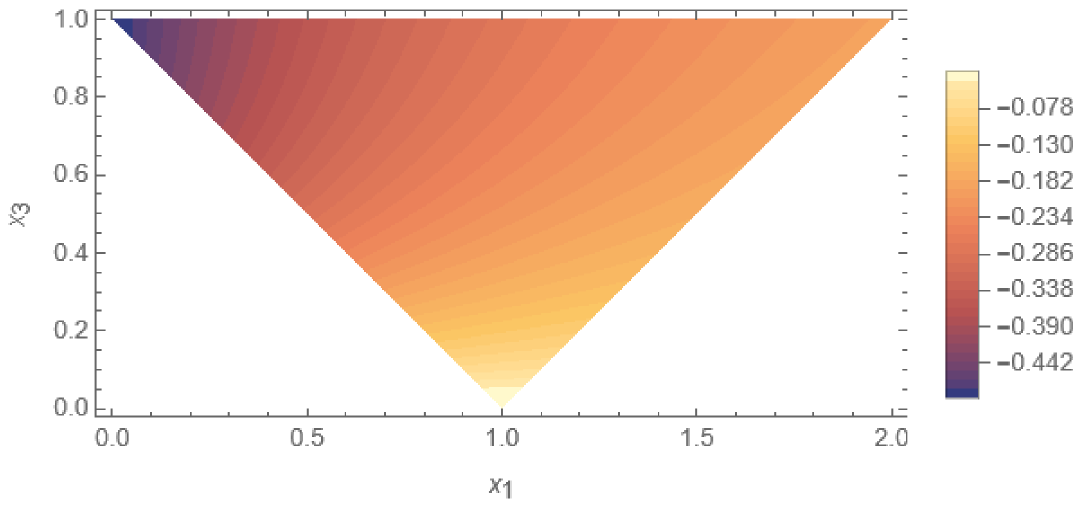


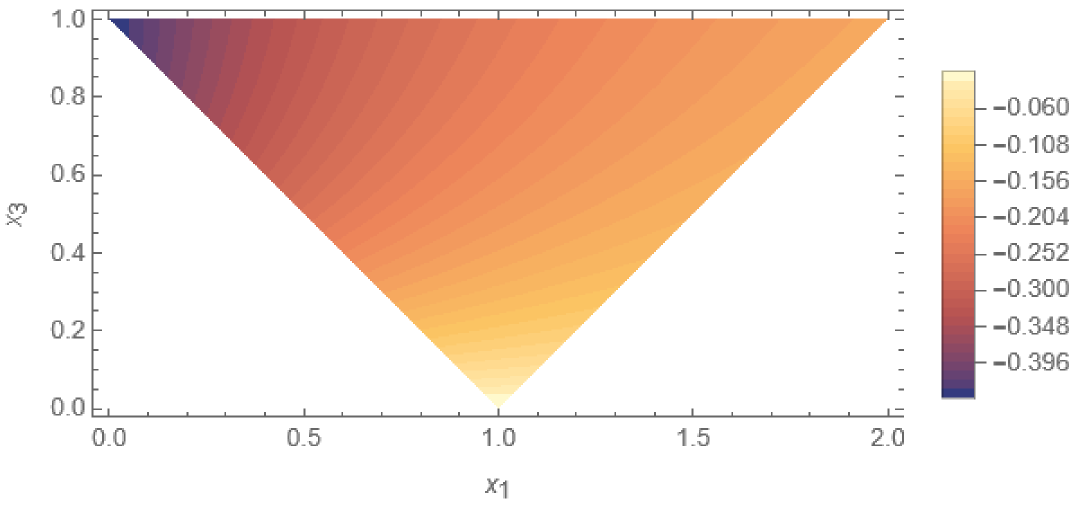


Plots of for

Axes:

Case II:





δ	Squeezed ($x_1 \to 0,\ x_3 \to 1$)	Flattened ($x_1 \to 2,\ x_3 \to 1$)	Equilateral ($x_1 \to 1,\ x_3 \to 1$)
0	-0.500	-0.375	-0.444
0.001	-0.499	-0.192	-0.281
0.05	-0.4500	-0.162	-0.245

Check Consistency Relations

- Scalar: $$\lim_{\mathbf{k}_1\to 0} \langle \zeta(\mathbf{k}_1,\tau_I)\sigma(\mathbf{k}_2,\tau_I)\sigma(\mathbf{k}_3,\tau_I)\rangle = \frac{\dot{\lambda}}{\lambda H}(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)P_\zeta(k_1)P_\sigma(k_2)$$
    - From Maldacena consistency relation, the factor should be (unless)
    - Violation because of super-horizon evolution of
- Tensor: $$\langle \gamma(\mathbf{k_1},\tau_I)\sigma(\mathbf{k_2},\tau_I)\sigma(\mathbf{k_3},\tau_I)\rangle_{\mathbf{k_1}\to 0} = \nu(2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\epsilon_{ij}\frac{k_{2,i}k_{2,j}}{k_2^2}\left(\frac{1+5\alpha H^2}{1+3\alpha H^2}\right)P_\gamma(k_1)P_\sigma(k_2)$$
    - Maldacena CR does not explain the factor
    - Violation of equivalence principle? Need to modify the background-wave approach?

Conclusions

- Explored the full bispectrum of cross-correlations between the spectator field and metric perturbations arising from non-minimal couplings using the in-in formalism.
- Constraints on the index in order to avoid power-law divergences in the super-horizon limit of three-point correlations: for , otherwise
- Effect of non-zero field mass and non-minimal coupling are observable in the equilateral and flattened limits, for
- Series approximation: leads to non-zero value of the non-correlated squeezed limit of , that should be zero from the background-wave approach. No such inconsistency for correlations
- Maldacena's approach does not capture the effects of the mass or the non-minimal coupling on the scalar cross-correlations in the squeezed limit.
- The full in-in result for tensor correlation is only consistent with Maldacena's approach in the limit , *i.e.*, in the absence of the derivative coupling.

Future Prospects

- Explore higher order correlations, *e.g.*, 4-point correlation functions: etc.
- Include higher-order diagrams
- Improve accuracy of series approximation
- Look for exact analytical integrals for
- Develop new background wave approach to capture the effect of mass term, non-minimal coupling, and derivative coupling.

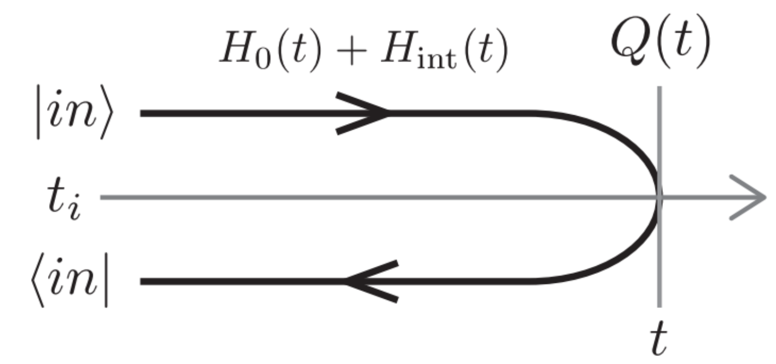
References

1. *Inflationary cross-correlations of a non-minimal spectator and their soft limits*, P. Jishnu Sai, Rajeev Kumar Jain, JCAP 09 (2023) 043
2. *Non-Gaussian features of primordial fluctuations in single field inflationary models*, Juan Martin Maldacena. JHEP, 05:013, 2003
3. *Planck 2018 results. IX. Constraints on primordial non-Gaussianity*. Astron. Astrophys., 641:A9, 2020
4. *On the non-Gaussian correlation of the primordial curvature perturbation with vector fields*, R. K. Jain and M. S. Sloth, JCAP 02 (2013) 003
5. *On non-Gaussianity in the curvaton scenario*. N. Bartolo, S. Matarrese, and A. Riotto. Phys. Rev. D, 69:043503, 2004
6. Useful lecture notes on the subject:
    1. *The Physics of Inflation*, ICTS Lecture notes, Daniel Baumann
    2. TASI Lectures on Inflation, Daniel Baumann

Thank you!

In-in Formalism

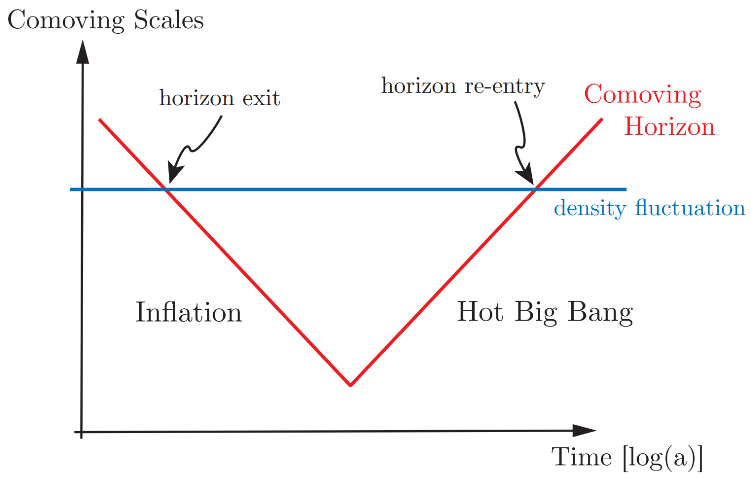
$$H = H_0 + H^{int}(t)$$



- Include interactions – perturbative solution
- Work in the interaction picture
- Compute expectation values in inflationary cosmology

- As opposed to *in-out* formalism (particle physics)

  - Out-state is not in Minkowski spacetime



Bunch-Davies Initial Condition

- Minkowski spacetime – vacuum is unique
- Inflationary epoch – vacuum state evolves with time.
- How to choose unique vacuum?
- Solution:
  - Look at sub-horizon dynamics. Mode functions approach harmonic oscillators
  - Deep sub-horizon regime is similar to Minkowski spacetime
  - Obtain lowest energy state in sub-horizon regime – define it as vacuum.
  - Bunch-Davies vacuum – ***in*** state of *in-in* formalism.

The Power Spectrum

- Position-space: Translation invariance
- Momentum-space: conservation of momentum

- Curvature Perturbations:

$$\Delta_\zeta^2(k) = \frac{k^3}{2\pi^2} P_\zeta(k)$$

(scale-independent power spectrum)

$$P_\zeta(k) \propto k^{n_s - 4}$$

$$n_\mathrm{s} - 1 = -2\varepsilon - \eta$$

The ‘Inflaton’

- Inflation: Needs shrinking horizon
- Matter/radiation: Expanding horizon
- Need new field: *Inflaton* field , coupled to gravity

$$S_{inflation} = \int d^4x \sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

$$\ddot{\phi} + 3H\dot{\phi} = -V'$$

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right]$$

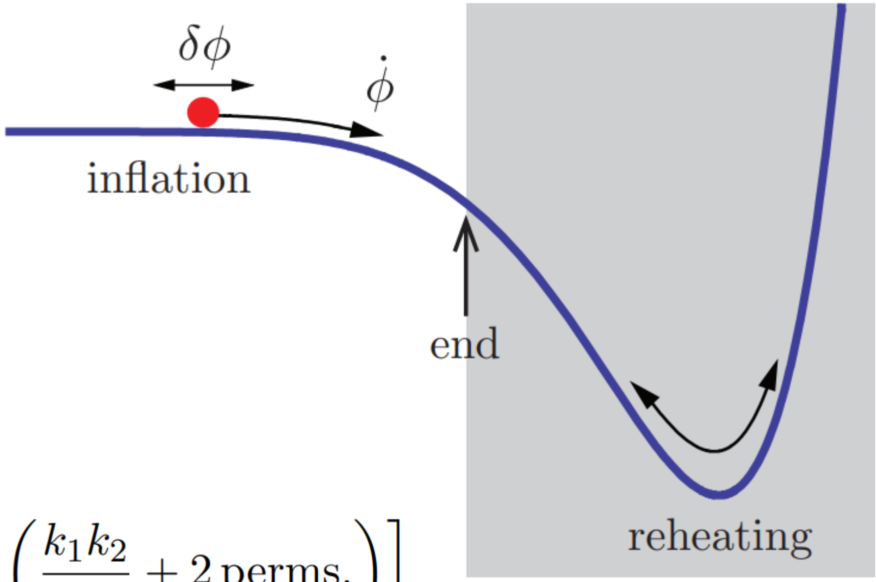
Slow-Roll Inflation

- Slow-roll parameters: $\varepsilon := -\frac{\dot{H}}{H^2}$ $\eta := \frac{\mathrm{d}\log\varepsilon}{\mathrm{d}\log a}$
- Slow-roll approximation: both
- Non-Gaussianities in slow-roll model: suppressed by slow-roll parameters

$$S_{\text{s.r.}} = \frac{11}{2}\varepsilon\,\mathcal{S}_\varepsilon + \frac{3}{2}\eta\,\mathcal{S}_\eta$$

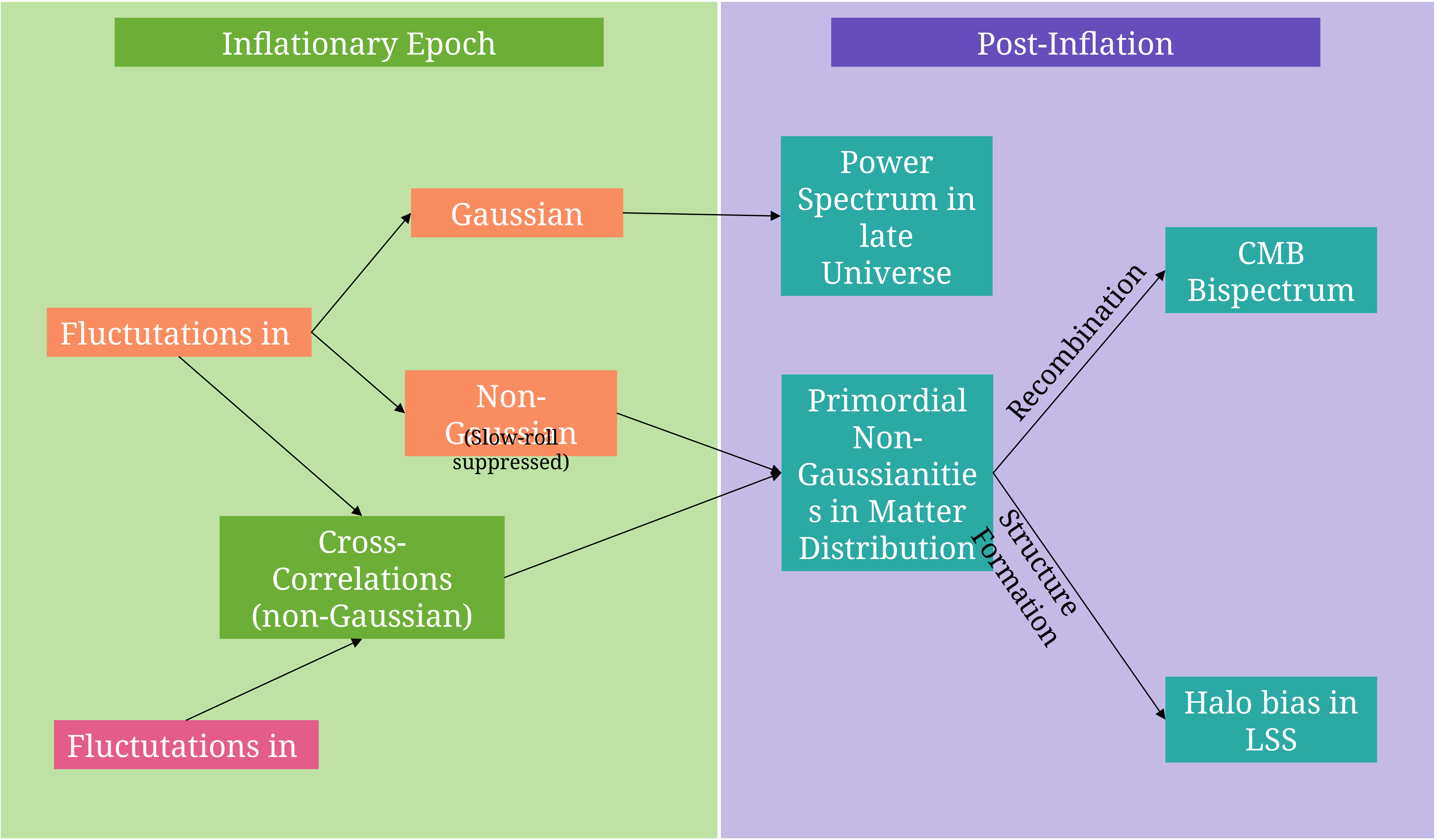
$$\mathcal{S}_\varepsilon = \frac{1}{11}\left[-\left(\frac{k_1^2}{k_2k_3} + 2\,\text{perms.}\right) + \left(\frac{k_1}{k_2} + 5\,\text{perms.}\right) + \frac{8}{3K}\left(\frac{k_1k_2}{k_3} + 2\,\text{perms.}\right)\right]$$

$$\mathcal{S}_\eta = \frac{1}{3}\left(\frac{k_1^2}{k_2k_3} + 2\,\text{perms.}\right),$$



Quasi de Sitter vs Perfect de Sitter

- Slow-roll:
- False vacuum inflation (perfect de Sitter):
  - Classical: Never ends
  - Quantum: Ends by tunnelling, but wrong inhomogeneities (too large/too small) – graceful exit problem



Inflationary Epoch
Post-Inflation
Gaussian
Power Spectrum in late Universe
CMB Bispectrum
Fluctutations in
Recombination
Non-Gaussian
(Slow-roll suppressed)
Primordial Non-Gaussianities in Matter Distribution
Structure Formation
Cross-Correlations (non-Gaussian)
Halo bias in LSS
Fluctutations in

Observational Status

- Observation of Non-Gaussianities:
  - CMB 3-point correlation function (Planck)
  - Halo bias in LSS (SDSS/DES)
- Planck 2018 results for (arXiv: astro-ph 1905.05697):

Shape	f_{NL}
Local	-0.9 ± 5.1
Equilateral	-26 ± 47
Orthogonal	-38 ± 24

- High uncertainty, expectant on future observations.

Non-Gaussianities in

Effect on large-scale structure: Halo bias

Effect on CMB anisotropies

All-sky surveys

Planck, WMAP

Free Theory of Perturbations

$$S_{\zeta\zeta} = \frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^3x\, a^2 \frac{\dot{\phi}^2}{H^2}\left(\dot{\zeta}^2 - (\partial_i\zeta)^2\right)$$

$$S_{\gamma\gamma} = \frac{1}{8}\int \mathrm{d}\tau \int \mathrm{d}^3x\, a^2\left[(\gamma'_{ij})^2 - (\partial_l\gamma_{ij})^2\right]$$

$$\zeta_g(\mathbf{k},\tau) = \hat{a}_{\mathbf{k}}\zeta_k(\tau) + \hat{a}^\dagger_{-\mathbf{k}}\zeta_k^*(\tau)$$

Divergences

- Superhorizon limit: End of inflation,
- Log-divergences: For example,
  - We usually take the limit for the horizon crossing
  - For log-divergences, take to get a ~60 factor in the bispectrum
- Power-law divergences: etc
  - Leads to enormous divergence factor in bispectrum ()

Dilatons

- Massless scalar fields
- Gauge-neutral, that is, no charges arising from gauge invariance.
- Inevitable in many UV-complete theories and string theories

Non-correlated squeezed limit

- Maldacena's approach:
- Similarly