On the Gravitational Stability of a Disk of Stars

Alar Toomre (1964)
Astrophys.J. 139 (1964) 1217-1238

Presentation as a part of
PH365: Galaxies and the Interstellar Medium

Presented by:
Souradeep Das
4th year Undergraduate
Indian Institute of Science

Date: 24 April 2024

Abstract

- Disk of stars initially rotating in balance between centrifugal and gravitational forces

- Small-scale disturbances are unstable when random velocities are absent
  - Critical length scale ~ size of disk
  - Growth time ~ period of revolution

- Sufficient velocity dispersion can eliminate instabilities at all scales

Outline

- Order of Magnitude estimate of stability criteria
- Analytical calculations in the absence of velocity dispersion
- Effect of velocity dispersion

Order-of-Magnitude Estimates

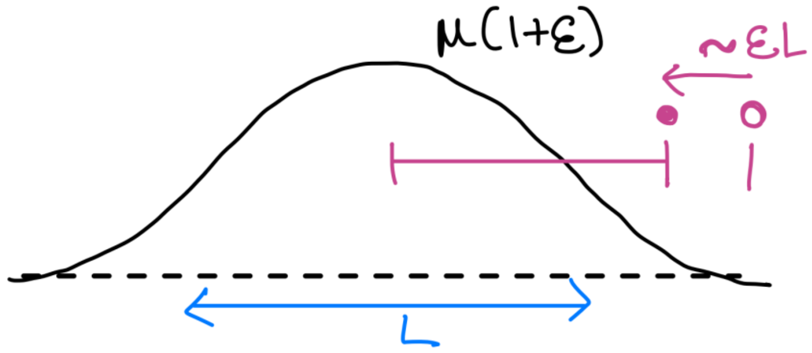
Rotation at Small Scales is Unstable

- Local rotation on disk
- Disturbance: length scale L, strength ε

- Particles move closer
- Stronger gravity $F_G \to F_G(1+\varepsilon)$

- Local rotation speeds up $\to \Omega_{local}\ (1+\varepsilon)$
- More centrifugal force $F_c \to F_c\left(1+\frac{3\varepsilon}{2}\right)$

- Stability when $F_C \gtrsim F_G$
- $L \gtrsim \frac{G\mu}{\Omega_{local}^2}$ is only stable… same order as disk size



Random Motion stabilizes Smaller Scales

- Random Motion causes diffusion

- Pre-existing disturbance at scale L, assume non-rotating disk
- Velocity dispersion leads to diffusion of particles over a timescale $\frac{L}{\sigma_u}$
- To maintain stability, this timescale must be less than the timescale of growth of disturbance $\left(\frac{L}{G\mu}\right)^{1/2}$

- Thus, $L \lesssim \frac{\sigma_u^2}{G\mu}$ for stability

- One can completely eliminate instability if $\frac{\sigma_u^2}{G\mu} \gtrsim \frac{G\mu}{\Omega_{local}^2}$, that is, $\sigma_u \gtrsim \frac{G\mu}{\Omega_{local}}$

Analytical Calculations

1. No Random Motions

Perturbations

$$v_r = u'(r, \theta, t)$$

$$v_\theta = V(r) + v'(r, \theta, t)$$

$$\mu = \mu(r) + \mu'(r, \theta, t)$$

$$\phi = \phi^0(r, z) + \phi'(r, z, \theta, t)$$

Linearized Equations of Motion

Newton's second law

$$\frac{\partial u'}{\partial t} + \Omega(r)\frac{\partial u'}{\partial \theta} - 2\Omega(r)\, v' = \left.\frac{\partial \phi'}{\partial r}\right|_{z=0}$$

(Potential$= -\phi$)

$$\frac{\partial v'}{\partial t} + \Omega(r)\frac{\partial v'}{\partial \theta} - 2B(r)\, u' = \left.\frac{1}{r}\frac{\partial \phi'}{\partial \theta}\right|_{z=0}$$

Continuity Equation

$$\frac{\partial \mu'}{\partial t} + \Omega(r)\frac{\partial \mu'}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}[r\mu(r)u'] + \frac{\mu(r)}{r}\frac{\partial v'}{\partial \theta} = 0$$

Poisson's Equation

$$\nabla^2\phi' = -4\pi G\mu'\delta(z)$$

Small-Wavelength Axisymmetric Disturbances

- Local radial waves around $r = r_0$:

$$\left.\begin{aligned} u'(r, t) &= C_1 \\ v'(r, t) &= C_2 \\ \mu'(r, t) &= C_3 \end{aligned}\right\} e^{iar}\, e^{i\omega t}$$

- Dispersion relation for wave

$$\boxed{2\pi G\mu(r_0)a = \kappa^2(r_0) - \omega^2},$$

$$\kappa^2(r_0) = -4\, B(r_0)\Omega(r_0)$$

- Critical wavelength (below which ω becomes imaginary: exponential solution – instability)

$$\boxed{\lambda_{\mathrm{crit}}(r_0) = 2\pi/a_{\mathrm{crit}} = 4\pi^2\, G\mu/\kappa^2}$$

- Timescale of growth

$$\tau = \kappa^{-1}[(\lambda_{\mathrm{crit}}/\lambda) - 1]^{-1/2}$$

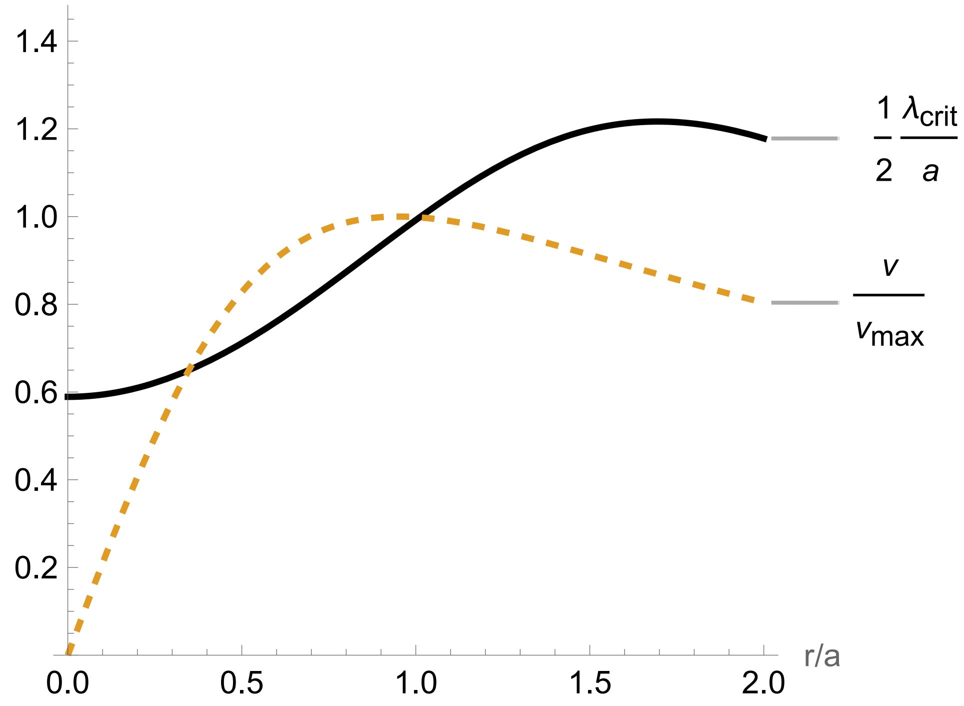
Galaxy Model

- Exact Model (Toomre, 1963)
- Rotational velocity curve:

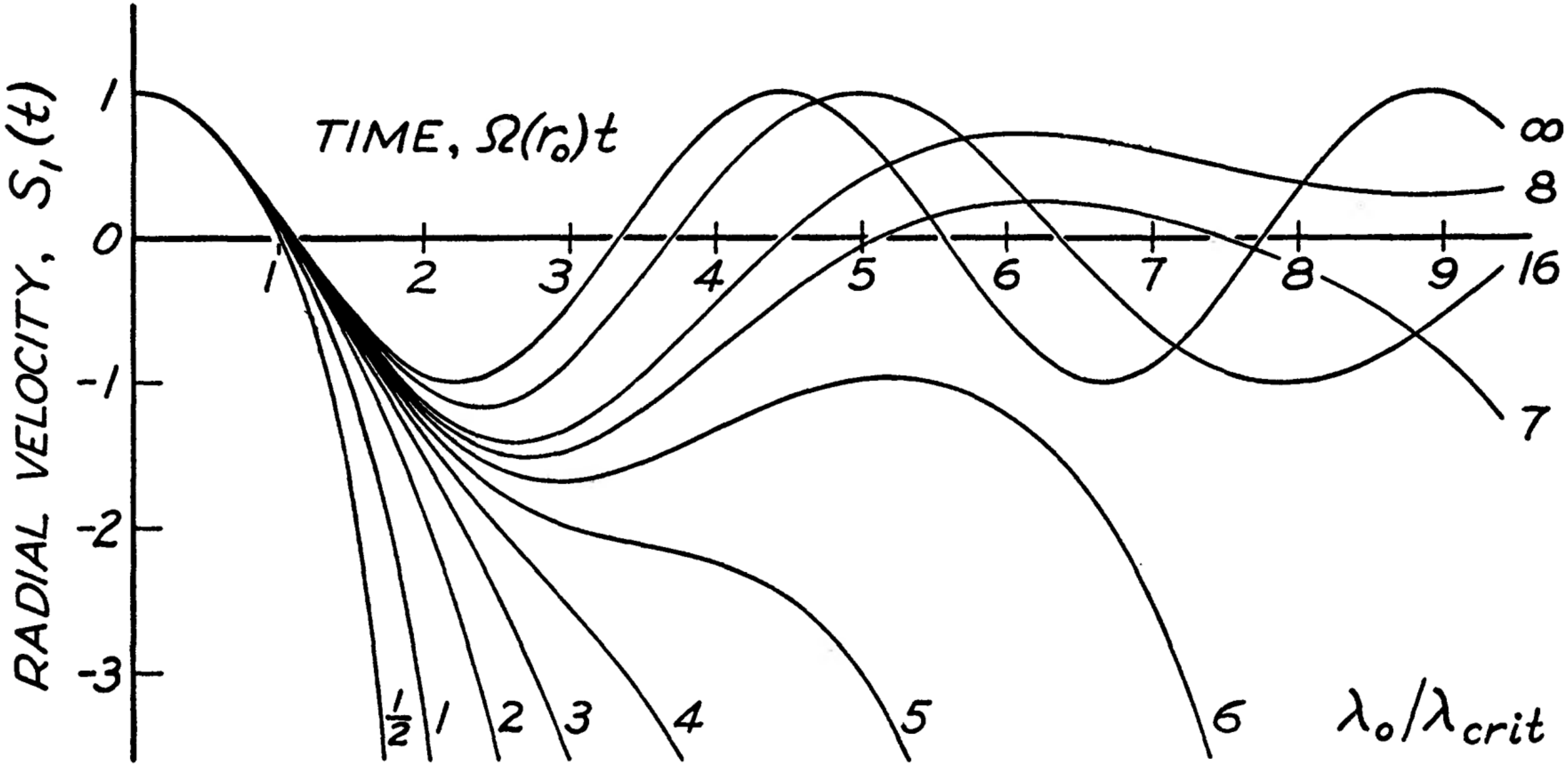
$$V(r) = \Omega_c r(1 + \tfrac{1}{4}r^2/a^2)^{1/2} (1 + r^2/a^2)^{-5/4}$$

- 2D density of mass:

$$\mu(r) = (3\Omega_c^2 a/8\pi G) (1 + r^2/a^2)^{-5/2}$$



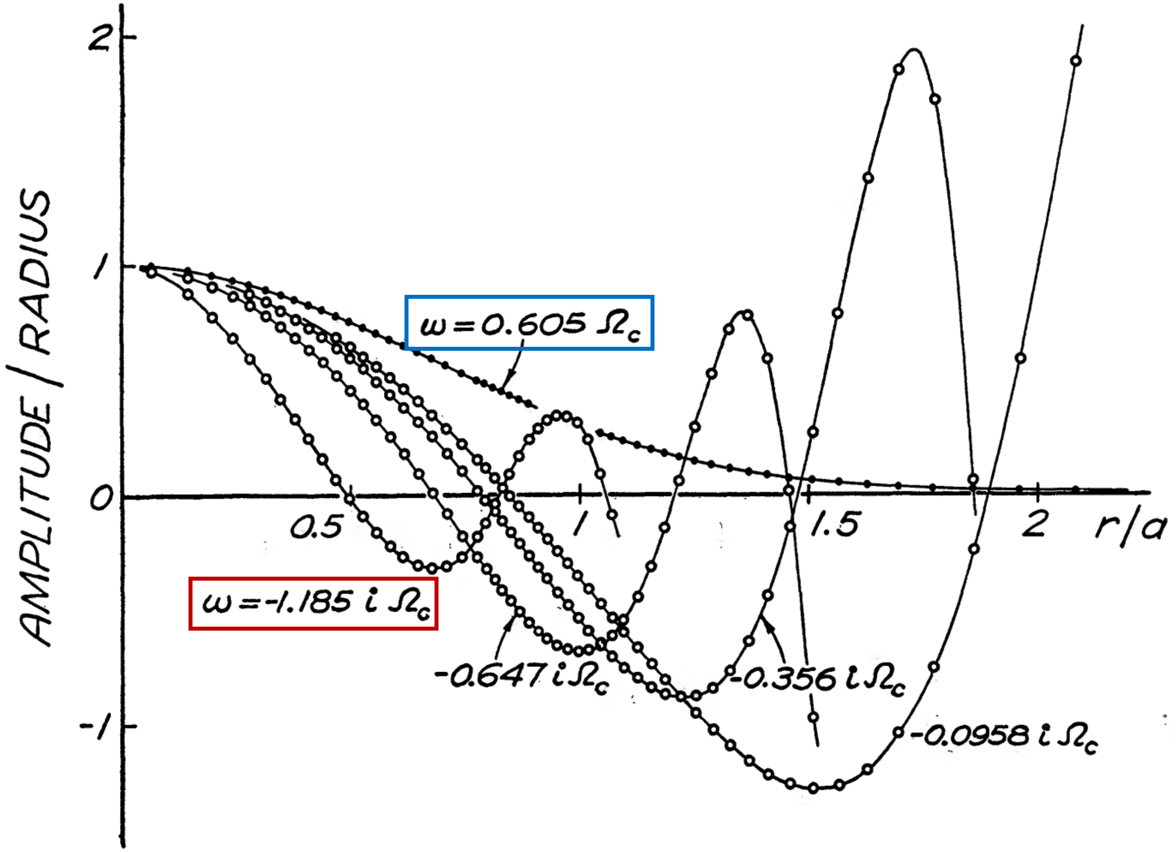
Non-Axisymmetric Disturbances



This example: Flat rotation curve, numerically integrated

Large-Wavelength Axisymmetric Disturbances

- Need numerical simulations.
- Smaller λ – larger imaginary ω – instability is worse



2. Effect of Random Motion

Collisionless Boltzmann Equation

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial r}+\frac{v^2}{r}\frac{\partial f}{\partial u}-\frac{uv}{r}\frac{\partial f}{\partial v}+F_r\frac{\partial f}{\partial u}=0$$

$$v=V(r)+w$$

Linearize:

$$f(u,v,r,t)=f^0(u,v,r)+f'(u,v,r,t)$$

$$\frac{\partial f'}{\partial t}+u\frac{\partial f'}{\partial r}+2w\Omega(r)\frac{\partial f'}{\partial u}-u\Omega(r)\frac{\partial f'}{\partial v}+F_r'\frac{\partial f^0}{du}=0$$

Solving Linearized CBE

- Assume zero-order distribution is a Gaussian

$$f^0(u, v, r) = (\mu/2\pi\sigma_u\sigma_v) \exp\,(-u^2/2\sigma_u{}^2 - w^2/2\sigma_v{}^2)$$

- First order distribution parametrized as:

$$f'(u, v, r, t) = f^0(u, w)\; g(u, w)\; e^{iar}\; e^{st}$$

- Solve PDE for $g(u, w)$ and use limit $s \to 0$ to get time-independent solutions. This is the condition for **neutral equilibrium** – boundary between stability and instability.

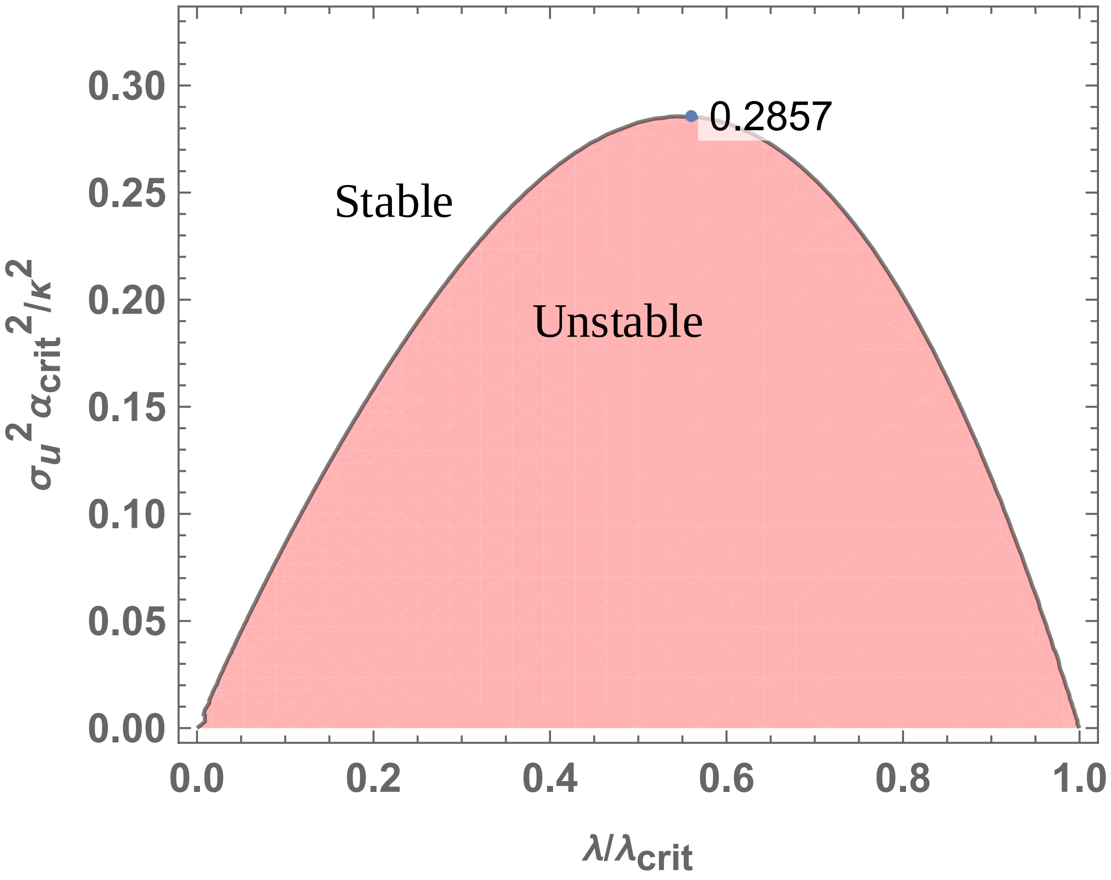
Stability Criterion

- **Critical Curve**– Neutral equilibrium
- Above the curve, axisymmetric disturbances are stable, below they are unstable
- In order to eliminate instability at all scales, one needs Y ≥ 0.2857 which gives $\sigma_u \geq 3.36\frac{G\mu}{\kappa}$
- σ_v is not independent of σ_u. Recall that:

$$\frac{\sigma_v^2}{\sigma_u^2} = -\frac{B}{\Omega}$$

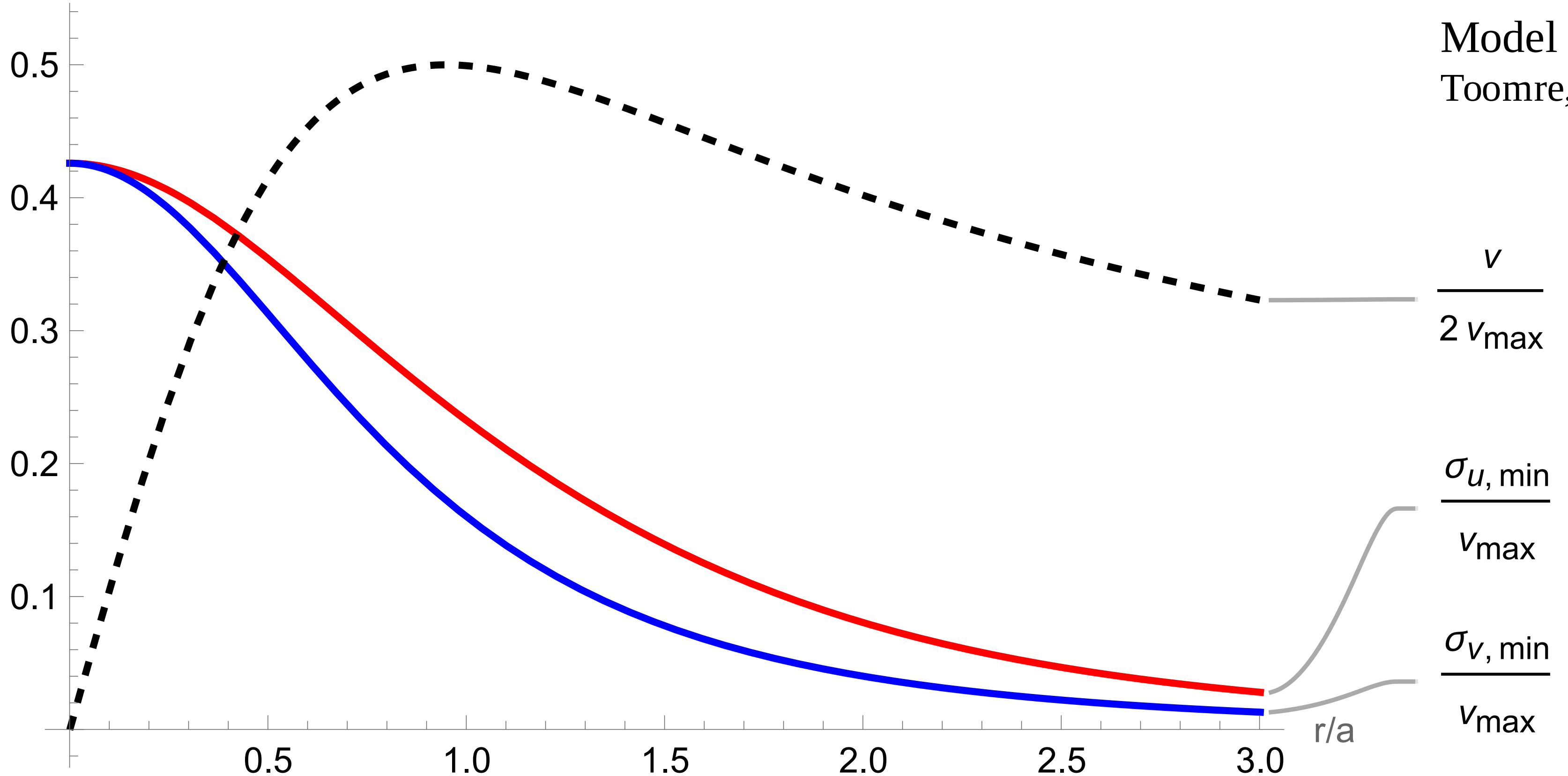
- In the limit of 0 dispersion, previous criteria are verified

$$\frac{\alpha_n \sigma_u{}^2}{2\pi G\mu} = 1 - \exp\left(-\sigma_u{}^2\alpha_n{}^2/\kappa^2\right) I_0(\sigma_u{}^2\alpha_n{}^2/\kappa^2)$$

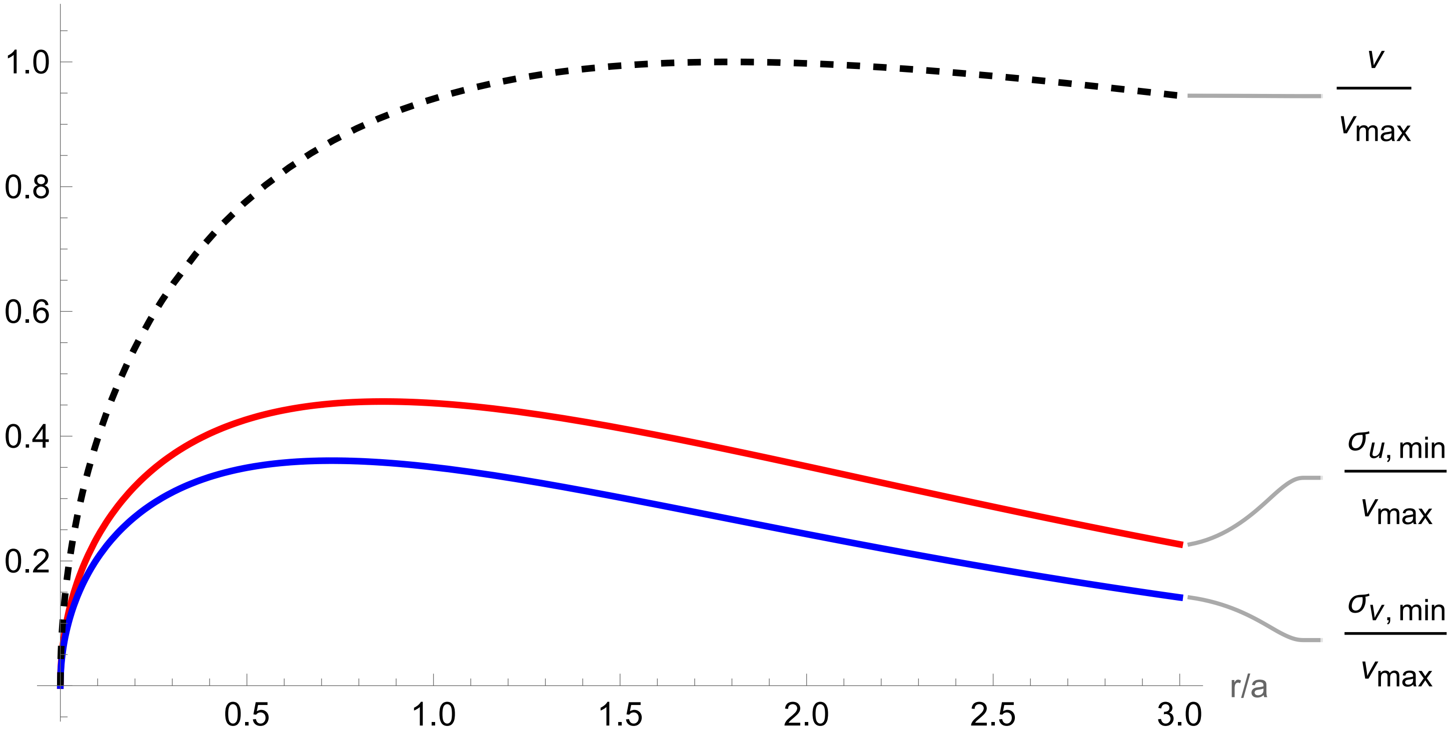


Minimum Dispersion to Avoid Instability

Model 2 defined in Toomre, 1963



Minimum Dispersion – Exponential Model



Exponential density profile

Conclusions

Summary

- Any stellar disk must be subject to instability
- Instabilities are vigorous – can be as large as disk size and grow as fast as rotation period

- To be stable, need large velocity dispersion – same order as rotation velocity.

- Did not consider dark matter halo.

Insights – Solar Neighbourhood

- $\mu = 50 - 65\ M_{\odot}/pc^2$
- $\kappa = 27 - 32\ km\ s^{-1} kpc^{-1}$
- $\lambda_{crit} = 8.5 - 15\ kpc$!!
- In the absence of random motions, instabilities of the galactic size are produced!

- Minimum required dispersion: $\sigma_u = 20 - 35\ km/s$ (Including effects of disk thickness, ISM)
- Consistent with most stars in solar neighbourhood

- Instabilities are not permanent – repeated cycles
- Results in more random motion - stability

Thank you!

Numerical Simulations

- N thin concentric rings of equal mass – placed accordingly

- Apply gravity between each pair (plus self-interaction)
- Radius and rotation velocity perturbed.
- Described by N linear equations – solve numerically (for N=80)