Neutrino Oscillations

A simple model using Quantum Mechanics

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Introduction

- Neutrinos are the lightest class of Fermions under the Standard Model of particle physics.
- They were first proposed by Wolfgang Pauli to resolve the apparent violation of energy conservation in β-decay.
- Neutrinos interact very weakly with matter, and are difficult to detect..

Figure: Wolfgang Pauli

Mass and Flavor

- In the Standard Model, neutrinos have zero mass, and hence travel at the speed of light.
- They exist in three flavors (ν_e, ν_μ, ν_τ), corresponding to the three families of leptons - Electrons, Muons and Tau. Each has its corresponding antineutrino ($\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$).

Solar Neutrino Problem

- The Sun produces a huge flux of neutrinos via the proton-proton chain.
- An experiment, called the **Homestake Experiment** was performed in the late 1960s under the leadership of Ray Davis and John Bahcall in a gold mine in the US, to detect and count solar neutrinos.
- The neutrino flux observed was only **one-third** of the theoretical prediction.
- This discrepancy could not be solved using the standard model of particle physics and the standard solar model, both of which were 'well-accepted'.

Solution to Solar Neutrino Problem

- A solution was devised based on a theoretical idea proposed first in 1957 by Soviet physicist Bruno Pontecorvo. This was the idea of **Neutrino Oscillations**.
- He proposed that neutrinos can transition between flavors spontaneously while traveling in free space.
- The Sun only produces electron neutrinos. These partially convert to the other flavors on the way to Earth. This resolves the discrepancy, because the experiment could only detect electron neutrinos.

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Flavor detection

- Neutrinos interact only via weak force and gravity.
- A neutrino is seen as a linear superposition of the three flavors.

$$|\nu\rangle = c_e |\nu_e\rangle + c_\mu |\nu_\mu\rangle + c_\tau |\nu_\tau\rangle \tag{1}$$

- When a neutrino is detected (through weak interaction with the detector), it randomly collapses to one of the three flavors. In other words, its flavor is 'measured'.
- The probability of detecting a particular flavor λ is given by $|c_\lambda|^2$ for $\lambda \in \{e, \mu, \tau\}$

Free neutrinos

- Neutrinos travel most of their way 'free', that is, devoid of interactions.
- The wavefunction in free state is a superposition of three plane waves, each corresponding to a different energy.
- Since neutrinos are ultra-relativistic,

$$E_i = \sqrt{p^2 + m_i^2} \approx p + \frac{m_i^2}{2p} \tag{2}$$

- Each plane wave has a slightly different mass, so the three waves have slightly different energies but the same 3-momentum.

Interlude: Wait a second!

- I just assumed that neutrinos have mass!
- According to the Standard Model, neutrinos are massless. So this model of oscillations will require a modification of the Standard Model.

Mass Eigenstates

- What if I tell you that the three plane wave eigenstates are not purely the three flavor eigenstates that I spoke about earlier?
- These are, in fact, a linear superposition of the three flavor states, with the coefficients given by the **Pontecorvo–Maki–Nakagawa–Sakata matrix** or **PMNS matrix**.
- These eigenstates are called mass eigenstates.

Relation between the two eigenstates

To put it simply, the two kinds of eigenstates are just two different bases of representing a neutrino wavefunction.

- When neutrinos travel freely, they it is convenient to represent them in the mass basis, since it is the eigenbasis of Hamiltonian for free plane waves.
- When neutrinos interact through the weak force, the Hamiltonian is diagonalized (in the flavor/mass subspace) by the flavor basis.

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The PMNS matrix

The PMNS matrix that describes the relation between mass and flavor eigenstates is defined as follows:
Say a wavefunction is written in the mass basis as:

$$|\Psi(t,\vec{x})\rangle \equiv \begin{pmatrix} c_1(t,\vec{x}) \\ c_2(t,\vec{x}) \\ c_3(t,\vec{x}) \end{pmatrix}_{(m)} \tag{3}$$

where the three neutrino masses are $m_1 < m_2 < m_3$.

The PMNS matrix

Similarly, it can be written in the flavor basis as:

$$|\Psi(t,\vec{x})\rangle \equiv \begin{pmatrix} c_e(t,\vec{x}) \\ c_\mu(t,\vec{x}) \\ c_\tau(t,\vec{x}) \end{pmatrix}_{(f)} \tag{4}$$

The PMNS matrix

Then the transformation between the bases is:

$$\begin{pmatrix} c_e(t,\vec{x}) \\ c_\mu(t,\vec{x}) \\ c_\tau(t,\vec{x}) \end{pmatrix} = \mathbf{U} \begin{pmatrix} c_1(t,\vec{x}) \\ c_2(t,\vec{x}) \\ c_3(t,\vec{x}) \end{pmatrix} \tag{5}$$

This 3×3 complex matrix $\mathbf{U}$ is called the PMNS matrix. It describes a rotation in complex space, and preserves the norm. So it must be a unitary matrix.

Parametrizing the matrix

The matrix is parametrized by 3 parameters:

- Mixing angles θ_{12}, θ_{23} and θ_{13}.
- A phase angle δ_{CP} related to charge-parity symmetry violation.

Note:

- This is one of the infinitely many ways to parametrize a unitary matrix.
- A $SU(3)$ matrix is described by 8 real parameters in general. But here, we can absorb the rest of the parameters into overall phases of the various wavefunction modes.

Parametrizing the matrix

Then we write:

$$\mathbf{U} = \mathbf{U_{23}}.\mathbf{U_{13}}.\mathbf{U_{12}} \tag{6}$$

where:

$$\mathbf{U_{23}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \quad \mathbf{U_{12}} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{U_{13}} = \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{+i\delta_{CP}} & 0 & \cos\theta_{13} \end{pmatrix}$$

On the parameters

- The mixing angles describe the extent of rotation around various axes. These have been estimated experimentally, and their latest values are:
  - $\theta_{12} = \left(33.41^{+0.75}_{-0.72}\right)^\circ$
  - $\theta_{23} = \left(49.1^{+1.0}_{-1.3}\right)^\circ$
  - $\theta_{13} = \left(8.54^{+0.11}_{-0.12}\right)^\circ$
- The phase represents deviation from charge-parity symmetry. If the symmetry were valid, we expect $\delta_{CP} = 0$. However, observations suggest $169^\circ \leq \delta_{CP} \leq 246^\circ$. The average value is:
  - $\delta_{CP} = \left(197^{+42}_{-25}\right)^\circ$
- This shows that CP symmetry is violated.

The Algorithm

1. Define the PMNS transformation $\mathbf{U}$ and obtain the inverse $\mathbf{U}^{-1}$.
2. Begin with an initial state vector in flavor basis.
3. Convert it to mass basis using $\mathbf{U}^{-1}$.
4. 'Evolve' the components of mass-basis vector independently using its energy eigenvalue upto some time t.
5. Transform the state ket to the flavor basis at time t using $\mathbf{U}$.

Important Trivia (Nice oxymoron!)

- The actual values of the masses m_1, m_2, m_3 do not matter. Rather, the differences in m_i^2 are what actually give rise to oscillations (in the low-mass limit, ofc).

Parameters used

I have used the mixing parameters from the best-fit model:

- $\theta_{12} = 33.41^\circ$
- $\theta_{23} = 49.1^\circ$
- $\theta_{13} = 8.54^\circ$
- $\delta_{CP} = 197^\circ$

For the other parameters I used:

- $m_1 = 0$ eV
- $\Delta m^2_{12} = 0.759 \times 10^{-4}$ eV^2
- $\Delta m^2_{13} = 23.2 \times 10^{-4}$ eV^2
- $p = 10^8$ eV $\approx E$

- **Important:** There are large uncertainties in the values of δ_{CP} and hence the model is not quantitatively too good.

Plotting oscillations

Start from a purely electron-neutrino and watch it oscillate!

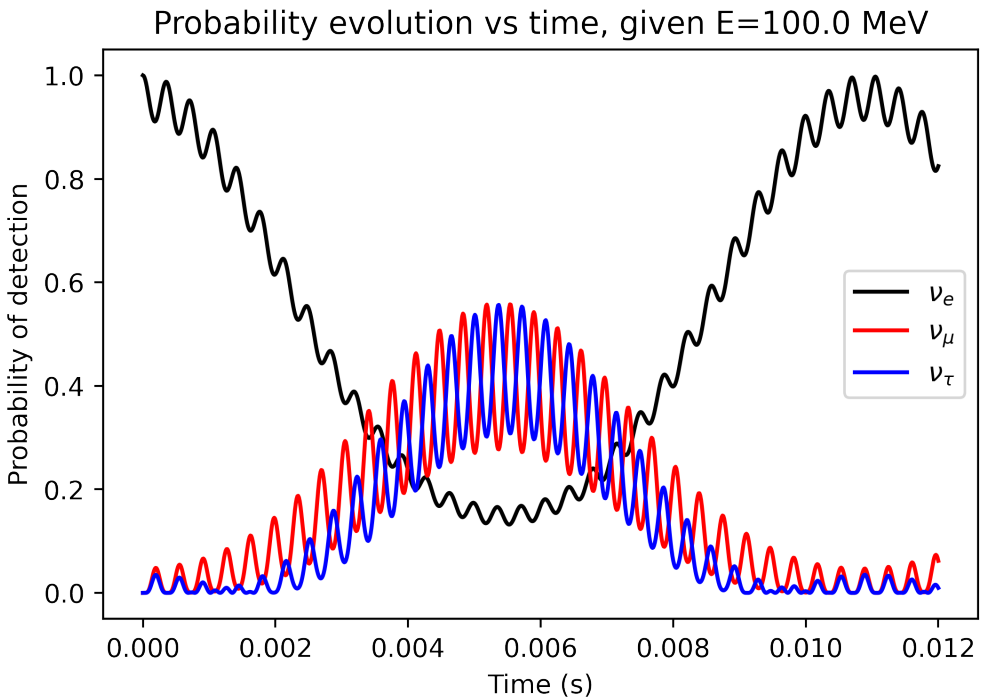

Figure: Oscillations of a state starting from pure ν_e

Plotting oscillations

It is an oscillation because it is periodic. Zoom out and see:

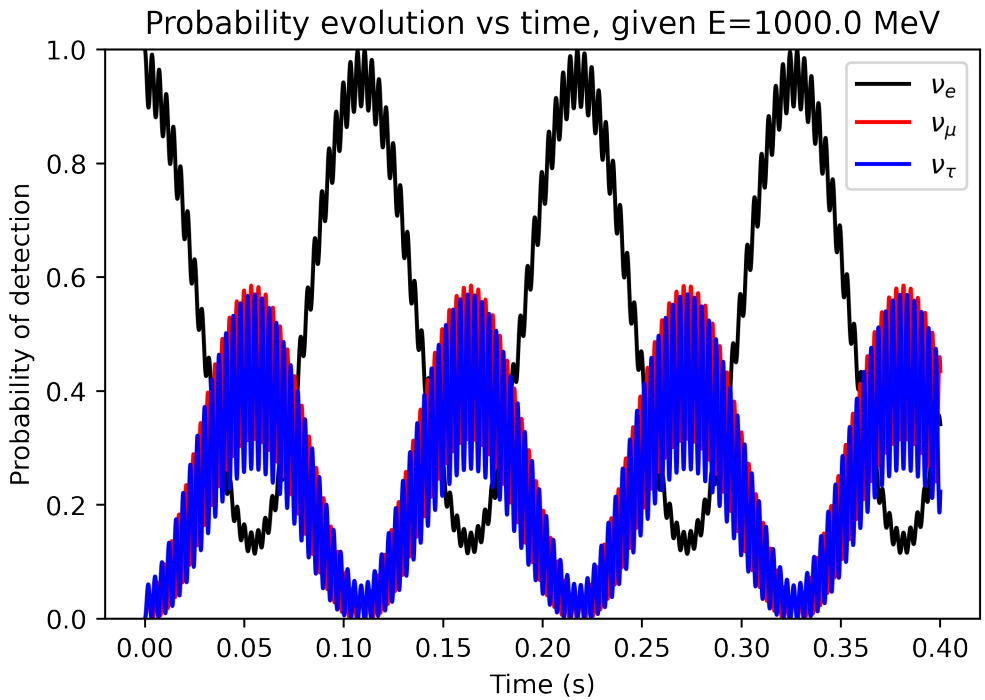

Figure: Oscillations of a state starting from pure ν_e

Plotting oscillations

The plot can be made more general if you consider $L/E = ct/E$ as the variable instead of time. The plot then becomes independent of E:

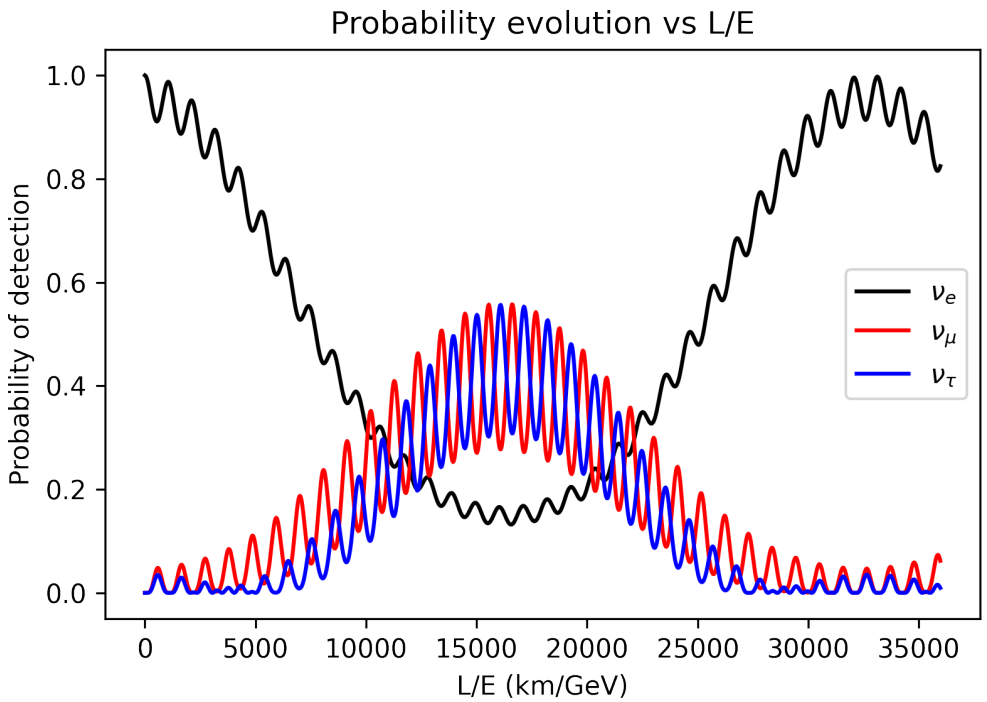

Figure: Oscillations of a state starting from pure ν_e

Plotting oscillations

Similar plots can be obtained starting from pure ν_μ or ν_τ:

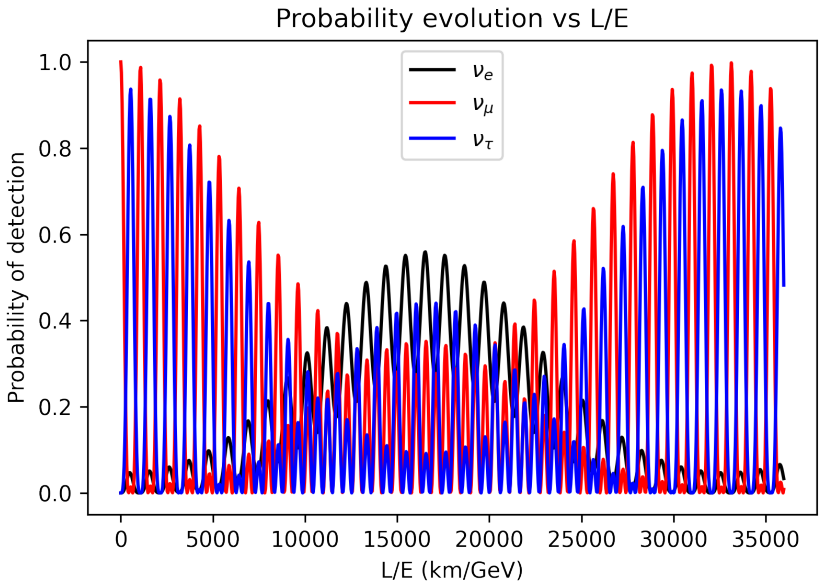

Figure: Starting from pure ν_μ

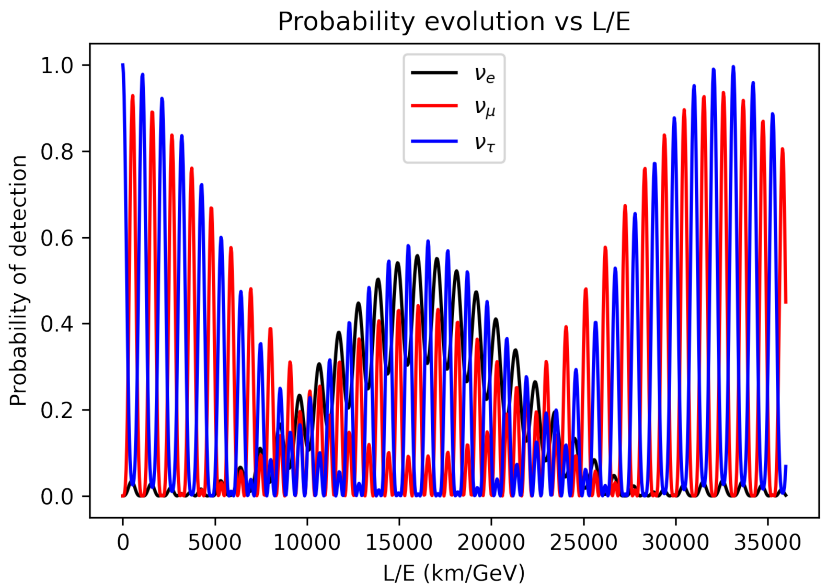

Figure: Starting from pure ν_τ

Did you observe something interesting?

- Did you see how so little of ν_μ or ν_τ were formed when the starting state was ν_e? Or vice-versa?

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Did you observe something interesting?

- Did you see how so little of ν_μ or ν_τ were formed when the starting state was ν_e? Or vice-versa?
- Did you also notice how the waveforms of ν_μ and ν_τ move together?
- Both of these are signs of stronger coupling between ν_μ and ν_τ, and weaker coupling of each with ν_e.

Beats

- Did you notice the waveforms? They consist of a large, low-frequency mode and a small, high-frequency mode.

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- Did you notice the waveforms? They consist of a large, low-frequency mode and a small, high-frequency mode.
- If you look carefully, there are nearly 30 small oscillations in a single large oscillation period.
- This 30 comes from the ratio of the mass-square differences,

$$\frac{\Delta m^2_{13}}{\Delta m^2_{12}} \approx 30$$

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Using Density Matrices

- For pure initial states, we can always use the wavefunction evolution as we saw before.
- But if the initial state is an ensemble of different flavors, the best tool to use is density matrices.
- Density matrix for a pure state $|\Psi\rangle$ is defined as $\hat{\rho} = |\Psi\rangle \langle\Psi|$.
- In general, density matrix exists for a mixed state as well.

Density Matrix Analogue

- Just as before, we can define an initial density matrix that keeps information about the distribution and correlations.
- The density matrices in the flavor and mass bases are related by:

$$\hat{\rho}_{(f)} = \mathbf{U}.\hat{\rho}_{(m)}.\mathbf{U}^{\dagger} \tag{7}$$

- This is obtained from the analogous rotation formula for the wavefunction:

$$|\Psi\rangle_{(f)} = \mathbf{U}\,|\Psi\rangle_{(m)} \tag{8}$$

Density Matrix Analogue

- The time-evolution of density matrix for a time-independent Hamiltonian is given by:

$$\hat{\rho}(t) = e^{-i\mathbf{H}t/\hbar}\hat{\rho}(0)e^{i\mathbf{H}t/\hbar} \tag{9}$$

The Algorithm

- The algorithm is quite similar to the wavefunction approach.
- Only differences are in the way we represent the state.

The Algorithm

1. Define the PMNS transformation $\mathbf{U}$ and obtain the inverse $\mathbf{U}^{-1}$.
2. Begin with an initial state density matrix in flavor basis.
3. Convert it to mass basis using $\hat{\rho}_{(m)} = \mathbf{U}^{\dagger}.\hat{\rho}_{(f)}.\mathbf{U}$.
4. 'Evolve' the components of mass-basis vector independently using its energy eigenvalue upto some time t.
5. Transform the density matrix to the flavor basis at time t using $\mathbf{U}$.

Interpretation of the density matrix

- In density matrix formalism, the expectation value of a quantity $\mathcal{O}$ associated with an operator $\hat{\mathcal{O}}$ is given by:

$$\left\langle \hat{\mathcal{O}} \right\rangle = \mathrm{Tr}\left(\hat{\rho}.\hat{\mathcal{O}}\right)$$

- The quantity whose expectation value gives the probability of the neutrino wave being in a flavor state $|\lambda\rangle$ is the projection operator $|\lambda\rangle\langle\lambda|$.
- So the projection operator for the electron flavor looks like:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the flavor basis.

Probabilities from the density matrix

- Hence it can be shown that the probabilities that we calculated in the wavefunction approach as $p(\lambda) = |c_\lambda|^2$ are just the diagonal elements of the density matrix in the flavor basis.
- So, after evolving the density matrix in the mass basis, we convert it again into the flavor basis and then read off the diagonal elements.

A closer look at the Evolution

- We could have evolved the density matrix in the flavor basis directly as well, by converting the Hamiltonian into this basis.
- However, it is easier to evolve in the mass basis because the Hamiltonian is diagonal, with elements $(p + \frac{m_1^2}{2p}, p + \frac{m_2^2}{2p}, p + \frac{m_3^2}{2p})$

Evolution in the mass basis

- The density matrix as time t is given by:

$$\hat{\rho}(t) = e^{-i\mathbf{H}t/\hbar}.\hat{\rho}(0).e^{+i\mathbf{H}t/\hbar} \tag{10}$$

- We can look at each element of the matrix in the mass basis:

$$\langle i|\,\hat{\rho}(t)\,|j\rangle = \langle i|\,e^{-i\mathbf{H}t/\hbar}.\hat{\rho}(0).e^{+i\mathbf{H}t/\hbar}\,|j\rangle$$

Evolution in the mass basis

- Now insert complete set of mass eigenstates between operators to get:

$$\begin{aligned}
\langle i|\,\hat{\rho}(t)\,|j\rangle &= \langle i|\,e^{-i\mathbf{H}t/\hbar}.\hat{\rho}(0).e^{+i\mathbf{H}t/\hbar}\,|j\rangle \\
&= \sum_{i',j'} \langle i|\,e^{-i\mathbf{H}t/\hbar}\,\left|i'\right\rangle \left\langle i'\right|\,\hat{\rho}(0)\,\left|j'\right\rangle \left\langle j'\right|\,e^{+i\mathbf{H}t/\hbar}\,|j\rangle \\
&= \sum_{i',j'} e^{-iE_it/\hbar} \left\langle i'\right|\,\hat{\rho}(0)\,\left|j'\right\rangle e^{+E_jt/\hbar}\delta_{ii'}\delta_{jj'} \\
\therefore\ \langle i|\,\hat{\rho}(t)\,|j\rangle &= \langle i|\,\hat{\rho}(0)\,|j\rangle\, e^{-i(E_i-E_j)t/\hbar}
\end{aligned}$$

Results

We can look at a mix of the flavors as an initial state, say in the ratio 1:1:1. In this special case, the state does not change physically with time. The initial density matrix is $\hat{\rho}(0) = \frac{1}{3}\mathbf{I}$ and remains so at all times.

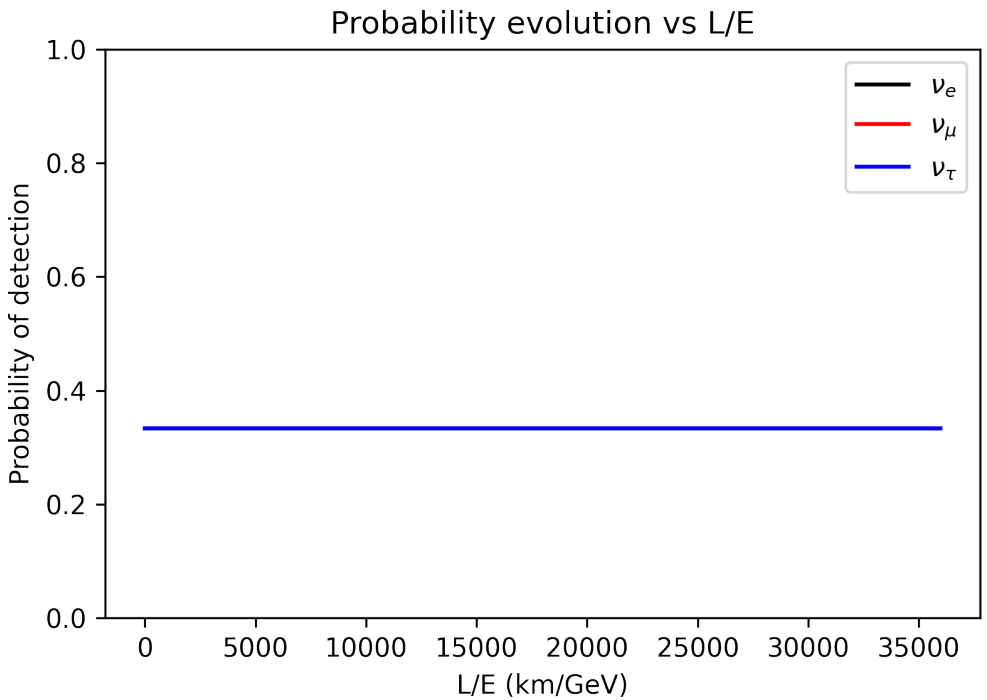

Figure: Oscillations of a state starting from an unpolarized flavor state

Results

Or we could begin from a state that is half-muon and half-tau, or half-electron and half-tau:

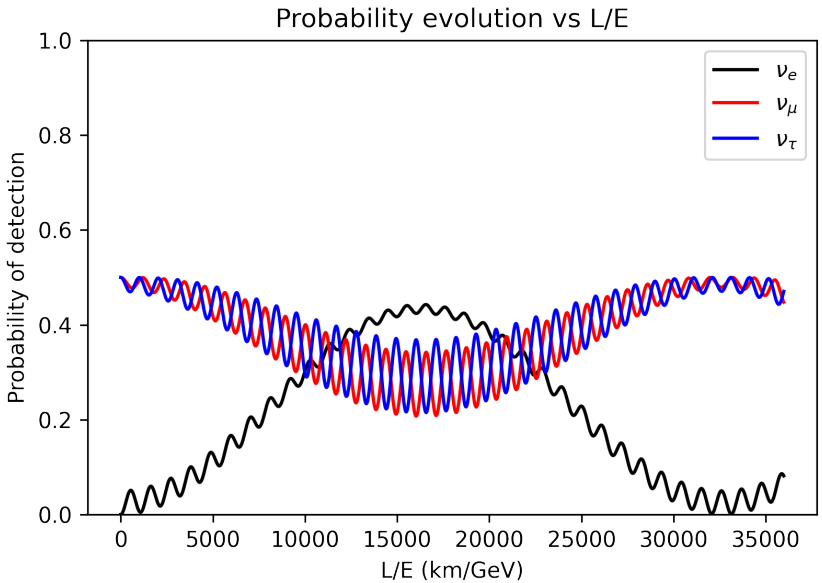

Figure: Starting from a mixed state of μ and τ

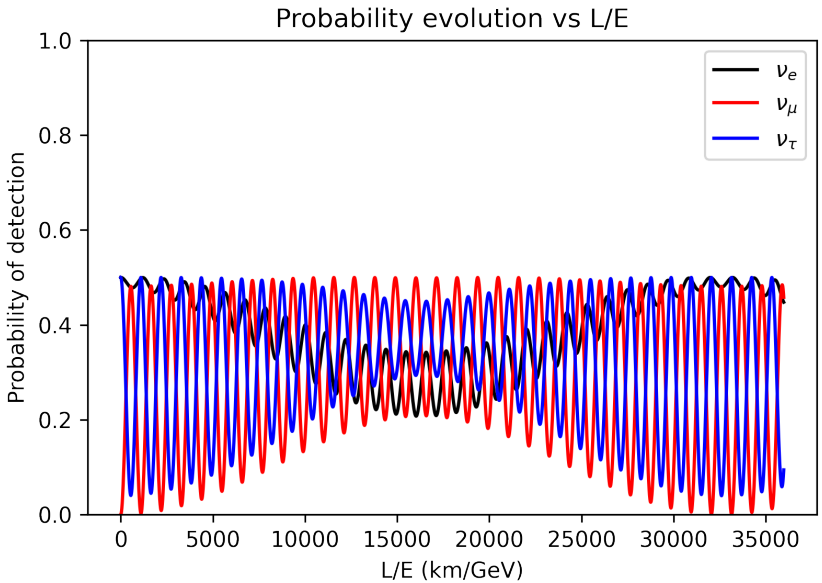

Figure: Starting from a mixed state of e and τ

Remarks

- Thus the density matrix approach is more general.

Correlations matter!

We can change the non-diagonal terms in the initial density matrix (keeping it Hermitian) which gives a different evolution pattern. For example, if the initial flavor density matrix is changed to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ the evolution of the state changes:

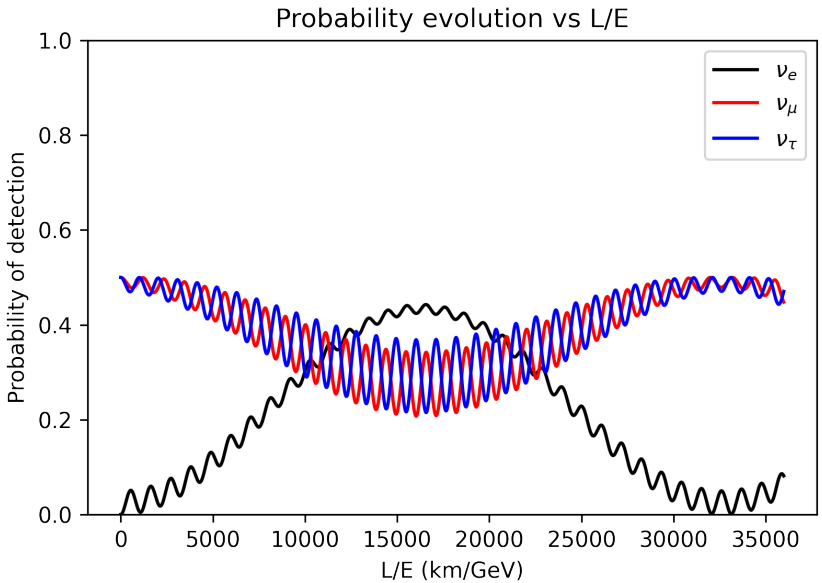

Figure: When the density matrix is diagonal

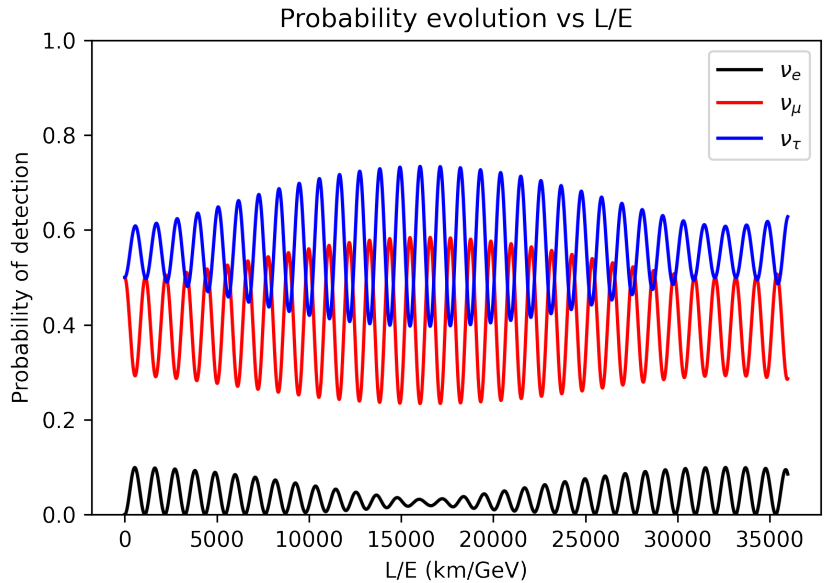

Figure: When the density matrix has off-diagonal terms

Code

The code can be found at:

`https://github.com/souradeepdas-iisc/neutrino-oscillations.git`